

Global Solutions to Compressible Navier-Stokes Equations With Spherically Symmetric Motion and Free Boundary

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Abstract

This work is devoted to study the global existence of strong and classical solutions to compressible Navier-Stokes equations with or without density jump on the moving boundary for spherically symmetric motion. We establish a unified method to track the propagation of regularity of strong and classical solutions which works for the cases when density connects to vacuum continuously and with a jump simultaneously. The result we obtain is able to deal with both strong solutions with physical vacuum for which the sound speed is $1/2$ -Hölder continuous across the boundary, and classical solutions with physical vacuum when $1 < \gamma < 3$. In contrast to the previous results of global weak solutions, we track the regularity globally-in-time up to the symmetric center and the moving boundary. In particular, the free boundary can be traced.

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1 Introduction

1.1 Description and Background

The motion of a viscous barotropic gas(or fluid) can be described by the isentropic compressible Navier-Stokes equations. In particular, the following system with constant viscosities governs the spherical motion in three dimensional space,

$$\begin{cases} \partial_t(r^2\rho) + \partial_r(r^2\rho u) = 0 & r \in (0, R(t)), \\ \partial_t(r^2\rho u) + \partial_r(r^2\rho u^2) + r^2\partial_r P = (2\mu + \lambda)r^2\partial_r \left(\frac{\partial_r(r^2u)}{r^2} \right) & r \in (0, R(t)), \end{cases} \quad (1.1)$$

where $\rho, u, R(t)$ represent the density, the radial velocity and the radius of the boundary respectively. The Lamé constants μ, λ denoting the viscosity coefficients would satisfy the relation $\mu > 0, 3\lambda + 2\mu \geq 0$. Moreover, the pressure potential P is assumed to depend only on the density. For simplicity in this work, the equation of state is taken as $P = \rho^\gamma$ with $\gamma > 1$. Also, we will work on the Navier-Stokes system (1.1) complemented with the following free boundary conditions,

$$\begin{aligned} [P - (2\mu r^2\partial_r u + \lambda\partial_r(r^2u))](R(t), t) &= 0, \\ u(0, t) &= 0, \quad \partial_t R(t) = u(R(t), t), \end{aligned} \quad (1.2)$$

where the first boundary condition represents the balance of stress tensor across the gas-vacuum interface. Also, the initial data is taken to be

$$R(0) = R_0, \quad u(r, 0) = u_0(r), \quad \rho(r, 0) = \rho_0(r), \quad r \in (0, R_0). \quad (1.3)$$

Without loss of generality, it is assumed $R_0 = 1$ in the following. Meanwhile, we do not impose any boundary profile on the initial density ρ_0 . In fact, ρ_0 can connect to the vacuum with or without a jump.

In particular, ρ_0 can approach the vacuum continuously across the boundary. Such gas-vacuum interface problem has appeared in plenty of physical scenarios such as astrophysics, shallow water waves etc. For example, the configuration of a non-rotating gaseous star would admit the physical vacuum boundary, i.e.

$$-\infty < \nabla_n c^2 \leq -C < 0, \rho = 0, \quad \text{on the boundary,} \quad (1.4)$$

where n denotes the normal direction and $c^2 = P'(\rho)$ is the square of the sound speed. Indeed, the physical vacuum boundary indicates that the sound speed c is only 1/2-Hölder continuous instead of Lipschitz continuous across the boundary, which is quite troublesome (see [18]). Only recently, some local-in-time well-posedness of the smooth solutions for such problems is available for the inviscid flows [4, 6, 5, 7, 8, 14, 16, 21] with or without self-gravitation and for the viscous flows [13] with self-gravitation. As for the global dynamic of flows with physical boundary (1.4), Luo, Xin, Zeng [23] have shown that with small perturbation of the Lane-Emden solutions, the strong solution to the Navier-Stokes-Poisson system exists globally and converges to the equilibrium state. See [22] for the case with degenerate viscosities. Meanwhile, Zeng has established the global regularity of the compressible Navier-Stokes equation in one dimensional setting which includes the case of physical vacuum in [42]. Zeng's work extends the one in [20], in which the authors have shown the global existence of the solutions to the one dimensional problem with constant viscosities but higher regularity for the density.

When the density connects to vacuum with a jump, a global weak solution with a spherically symmetric motion to the problem with density-dependent viscosities is obtained in [9] by Guo, Li, Xin. Moreover the solution is shown to be smooth away from the centre. Recently, such problem is studied in the setting of spherically symmetry in two dimensional space and $\mu = \text{constant}$, $\lambda = \lambda(\rho) = \rho^\beta$ with some $\beta > 1$ by Li, Zhang [17]. Working in both Lagrangian and Eulerian coordinates, the authors show the global existence of strong solutions. Another noticeable result is from Yeung and Yuen [35], in which the authors have constructed analytic solutions in the case with density-dependent viscosities. Similar results were further studied in [10] with or without a density jump across the boundary. Such solutions indicate that the domain of the gas(fluid) would expand as time grows up, and the density would decreases to zero everywhere including the centre.

Notice, for a spherically symmetric motion, the regularity of solutions at the centre has been only obtained in the case when it is with small moves, induced either by the local-in-time motion [21] or by the small perturbation and stability of the equilibrium state [23, 22]. However, when such structures do not exist, it is in general not clear how to perform similar estimates. In this work, we consider the global regularity of (1.1) with the free boundary (1.2). For one thing, in the case when the density connects to vacuum continuously, the degeneracy of the system makes the problem challenging as mentioned in [13]. Indeed, the classical techniques [36, 37, 40, 39] does not work in such a situation. Besides the degeneracy on the boundary, we will focus on the regularity at the centre. As mentioned above, the moving domain is expected to expand due to the absence of equilibrium state and large time. In particular, there is no a priori bound on the flow trajectories which causes additional difficulties.

Before moving onto our strategy to overcome the difficulties listed above, we briefly review some classical works concerning the compressible Navier-Stokes system.

There are rich literatures studying the Cauchy and first initial boundary value problem. In the absence of vacuum ($\rho \geq \underline{\rho} > 0$), the local and global well-posedness of classical solutions have been investigated widely. To name a few, Serrin [29] considered the uniqueness of both viscous and inviscid compressible flows. Itaya [12] and Tani [31] considered the Cauchy and first initial boundary problems. See also [32]. Moreover, the pioneering works of Matsumura and Nishida [24, 25] showed the global well-posedness of classical solutions to Navier-Stokes equations with small perturbation of a uniform non-vacuum state. When the vacuum appears, some singular behaviour may occur. For instant, as pointed out by Xin in [33], the classical solution to the Navier-Stokes equations may blow up in finite time. See also in [34, 1]. Nevertheless, the local well-posedness theory was developed by Cho and Kim [3, 2] for barotropic and heat-conductive flows. See also [43]. Unfortunately, the solutions from Cho and Kim is not in the same functional space as that in [33]. In particular, it can not track the entropy in the vacuum area. With small initial energy, Huang, Li, Xin [11] show the global existence of classical solutions but with large oscillations to the isentropic compressible Navier-Stokes equations. However, as pointed out in [19], such Navier-Stokes system might not develop a satisfactory solution at the vacuum state. Therefore, the authors introduced the problem with density-dependent viscosities and showed the well-posedness of the free boundary problem locally.

As for the free boundary problems, the local well-posedness theory can be tracked back to Solonnikov and Tani [30], Zadrzyńska and Zajackowski [36, 38, 39, 41]. The global well-posedness problem were studied in [37, 39, 40]. Among these works, the stress tensor on the moving surface is balanced by a force induced by the surface tension or an external pressure. In particular, it admits a uniform state and the global well-posedness was achieved with small perturbation of such constant state. We refer other free boundary problems to [45, 15, 26, 27, 28, 44] and the references therein.

We will work in the Lagrangian coordinates induced by the flow trajectories, defined in (1.5). Comparing to the classical choice of Lagrangian mass coordinates, such coordinates would enable us to track the particle path both at the centre and near the moving surface as mentioned in [42]. Indeed, as it will explain itself, the Lagrangian unknown r has ready represented the density ρ and the velocity u (1.6), and ρ_0 would appear as a degenerate coefficient as in [21]. The coordinate singularity at the center can be understood as follows. Let $\mathcal{U} = (V_1, V_2, V_3) = V(r) \cdot \vec{y}/r \in \mathbb{R}^3$ be a smooth radial vector field in \mathbb{R}^3 where $r = |\vec{y}|$. Then $|\nabla \mathcal{U}|^2 = V'^2 + 2(\frac{V}{r})^2$. In the Lagrangian coordinates system, such quantity is $(\frac{V_x}{r_x})^2 + 2(\frac{V}{r})^2$. Thus it can be seen, in order to obtain the desirable bound on the vector field $v(x, t) = u(r(x, t), t)$, the reasonable quantities to consider are $\frac{v_x}{r_x}$ and $\frac{v}{r}$. Such structure has already been studied in [21, 22, 23]. In the meantime, $\frac{r}{x}$ a priori admits only lower bound but not upper bound in contrast to the works by Luo, Xin, Zeng. In other word, we only expect the uniform (in time) bound of $\frac{x}{r}$ and $\frac{1}{r_x}$. While in [42], for the one dimensional problem, r_x can be estimated through a careful point-wise estimate, similar structure no longer exists for the spherical motion. In fact, the corresponding quantity is now $\frac{r^2 r_x}{x^2}$. However, the structure for this new quantity is not determinant enough (see (2.23)). In particular, it contains an extra boundary term $\frac{v}{r}|_{x=1}$. Luckily, the radius $R(t)$ priori admits uniform (in time) lower bound. Such structure would enable a $L_t^2 L_x^\infty$ estimates of $\frac{v_x}{r_x}$ and $\frac{v}{r}$. Our $L_t^\infty L_x^\infty$ estimates are achieved through some point-wise estimates, which are performed by integrating the equation from the boundary and from the centre. In fact, such calculations separate $\frac{v_x}{r_x}$ and $\frac{v}{r}$ from the viscosity tensor (see (2.42) and (2.44)). The benefit of such programs is that it avoids the integral multipliers used in [23] and the Hardy's inequality in [42], and therefore we are able to manipulate the case when ρ_0 is with general profiles. Eventually, the bound of $\frac{v_x}{r_x}, \frac{v}{r}, \frac{x^2}{r^2 r_x}$ can be represented by the a priori bound (2.1) and the initial energy $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$ defined in (1.12). Then by choosing initial energy small, we shall close the a priori

estimate. In addition, we study the relation between the a prior bound and the initial energy, from which it is possible to determine the a prior bound as a map of the initial energy, provided the initial energy small enough. We further discuss the regularity propagated by the solution, which would give the global regularity.

The rest of the present work would be organised as follows. In the next section, we would introduce the basic notations and the main results in the Lagrangian coordinates. Then we shall start the a priori estimates. Under the a prior assumption, some standard energy estimates would be listed in Section 2.1. The key estimates are studied in Section 2.2. We establish the uniform point-wise estimates discussed above. Furthermore, the smallness constraints on the initial energy is designed to close the estimates. Section 2.3 and Section 2.4 are devoted to study the propagation of regularity for strong and classical solutions respectively. In the end, we present the suitable functional frame-work for the local well-posedness problem and sketch the proof of the main theorem.

1.2 Lagrangian Reformulation and Main Results

We define the Lagrangian Coordinate as follow. For $x \in (0, 1) = (0, R_0)$, let the particle path r be a function defined by the following ordinary differential equation,

$$\begin{cases} \frac{d}{dt}r(x, t) = u(r(x, t), t), \\ r(x, 0) = x. \end{cases} \quad (1.5)$$

The Lagrangian unknowns are denoted by

$$f(x, t) := \rho(r(x, t), t), \quad v(x, t) := u(r(x, t), t).$$

Then the system (1.1) can be written as

$$\begin{cases} (r^2 f)_t + (r^2 f) \cdot \frac{v_x}{r_x} = 0, \\ r^2 f v_t + r^2 \frac{P_x}{r_x} = (2\mu + \lambda) \frac{r^2}{r_x} \partial_x \left(\frac{(r^2 v)_x}{r^2 r_x} \right). \end{cases}$$

Or equivalently,

$$\rho(r(x, t), t) = f(x, t) = \frac{x^2 \rho_0}{r^2 r_x}, \quad u(r(x, t), t) = v(x, t) = \partial_t r, \quad (1.6)$$

$$\left(\frac{x}{r}\right)^2 \rho_0 v_t + \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_x = (2\mu + \lambda) \left[\frac{(r^2 v)_x}{r^2 r_x} \right]_x = \mathfrak{B}_x + 4\mu \left(\frac{v}{r} \right)_x, \quad (1.7)$$

where

$$P = \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma, \quad \mathfrak{B} = (2\mu + \lambda) \frac{v_x}{r_x} + 2\lambda \frac{v}{r}.$$

(1.7) is complemented with the boundary condition

$$\begin{aligned} \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma - \left((2\mu + \lambda) \frac{v_x}{r_x} + 2\lambda \frac{v}{r} \right) \Big|_{x=1} &= 0, \\ r|_{x=0} &= 0, \quad v|_{x=1} = 0, \end{aligned} \quad (1.8)$$

and initial data

$$r(x, 0) = x, \quad r_t(x, 0) = u_0(x). \quad (1.9)$$

Moreover, the following assumption is imposed on the initial density ρ_0 , and the viscosity coefficients μ, λ ,

$$\rho_0 > 0 \text{ for } x \in [0, 1), \quad \max_{0 \leq x \leq 1} \rho_0 \leq \bar{\rho}_0, \quad (1.10)$$

$$\|(\rho_0^\gamma)_x\|_{L_x^2(0,1)} < \infty, \quad \mu, \lambda > 0. \quad (1.11)$$

Now we define the initial energy we shall use in this work. Denote

$$\begin{aligned} \mathcal{E}_0 &= \frac{1}{2} \int_0^1 x^2 \rho_0 u_0^2 dx + \frac{1}{\gamma - 1} \int_0^1 x^2 \rho_0^\gamma dx \\ \mathcal{E}_1 &= \frac{1}{2} \int_0^1 x^2 \rho_0 u_1^2 dx, \quad \mathcal{E}_2 = \frac{1}{2} \int_0^1 \rho_0 u_1^2 dx \\ \mathcal{E}_3 &= \frac{1}{2} \int_0^1 x^2 \rho_0 u_2^2 dx, \quad \mathcal{E}_4 = \frac{1}{2} \int_0^1 \rho_0 u_2^2 dx. \end{aligned} \quad (1.12)$$

where

$$u_1 = \frac{1}{\rho_0} \left\{ (2\mu + \lambda) \left(\frac{(x^2 u_0)_x}{x^2} \right)_x - (\rho_0^\gamma)_x \right\}, \quad (1.13)$$

$$\begin{aligned} u_2 &= \frac{1}{\rho_0} \left\{ (2\mu + \lambda) \left[\frac{(x^2 u_1)_x}{x^2} \right]_x + \gamma \left[\rho_0^\gamma \frac{(x^2 u_0)_x}{x^2} \right]_x \right. \\ &\quad \left. - (2\mu + \lambda) \left[u_{0,x}^2 + 2 \frac{u_0^2}{x^2} \right] \right\} + 2 \frac{u_0 u_1}{x}. \end{aligned} \quad (1.14)$$

Also, let $M > 0$ be a constant such that

$$\max_{0 \leq x \leq 1} \left\{ \left| \frac{u_0}{x} \right|, |u_{0x}| \right\} \leq M, \quad (1.15)$$

and the following compatible condition on the boundary is imposed,

$$\rho_0^\gamma - \left((2\mu + \lambda)u_{0,x} + 2\lambda \frac{u_0}{x} \right) \Big|_{x=1} = 0. \quad (1.16)$$

Then our main theorem is stated as follows

Theorem 1.1 (Global existence) *Consider the initial boundary problem (1.7), (1.8), (1.9) satisfying the assumption (1.10), (1.11), (1.15), (1.16), There is a $\bar{\epsilon} > 0$ depending on $\bar{\rho}_0, \mu, \lambda, M$ such that if*

$$\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 < \bar{\epsilon} \quad (1.17)$$

it admits a globally defined strong solution $(r(x, t), v(x, t))$ to (1.7). Moreover, there are positive constants $\bar{\alpha}$ and $\bar{\beta} > M$ such that

$$0 < \frac{x^2}{r^2 r_x} < \bar{\alpha}^3, \quad 0 \leq \left| \frac{v}{r} \right|, \left| \frac{v_x}{r_x} \right| < \bar{\beta}. \quad (1.18)$$

And the following regularities hold for any $0 < T < \infty$,

$$\begin{cases} x\sqrt{\rho_0}v, x\sqrt{\rho_0}v_t, \sqrt{\rho_0}v_t \in L_t^\infty((0, T), L_x^2(0, 1)), \\ v, v_x, v_{xx}, r, r_x, r_{xx}, \frac{v}{x}, \left(\frac{v}{x}\right)_x, \left(\frac{r}{x}\right)_x \in L_t^\infty((0, T), L_x^2(0, 1)), \\ xv_x, xv_{xt}, v, v_t, \frac{v_t}{x}, v_{xt} \in L_t^2((0, T), L_x^2(0, 1)). \end{cases} \quad (1.19)$$

If, in addition,

$$\mathcal{E}_3, \mathcal{E}_4 < \infty, \quad \|(\rho_0)_x\|_{L_x^2(0, 1)}, \|(\rho_0^\gamma)_{xx}\|_{L_x^2(0, 1)} < \infty, \quad (1.20)$$

the global strong solution is smooth, and satisfies, along with (1.19)

$$\begin{cases} x\sqrt{\rho_0}v_{tt}, \sqrt{\rho_0}v_{tt}, v_{xxt}, \left(\frac{v_t}{x}\right)_x \in L_t^\infty((0, T), L_x^2(0, 1)), \\ r_{xxx}, \left(\frac{r}{x}\right)_{xx}, v_{xxx}, \left(\frac{v}{x}\right)_{xx} \in L_t^\infty((0, T), L_x^2(0, 1)), \\ xv_{xtt}, v_{tt}, v_{xtt}, \frac{v_{tt}}{x} \in L_t^2((0, T), L_x^2(0, 1)), \end{cases} \quad (1.21)$$

for $0 < T < \infty$.

2 A Prior Estimates

Through this section, it is assumed $(r, v) = (r(x, t), v(x, t))$ is a smooth solution to (1.7) with (1.8), (1.9) such that all the integrations by parts in the following hold. Moreover, it is a priori assumed

$$0 < \frac{x^2}{r^2 r_x} \leq \alpha^3, \quad 0 \leq \left| \frac{v}{r} \right|, \left| \frac{v_x}{r_x} \right| \leq \beta \quad (2.1)$$

for some $\alpha > 1$ and $\beta \geq M$ (M is defined in (1.15)). From (2.1) it can be derived

$$r^3 = 3 \int_0^x r^2 r_x dx \geq 3\alpha^{-3} \int_0^x x^2 dx = \alpha^{-3} x^3, \quad \text{or } 0 < \frac{x}{r} \leq \alpha. \quad (2.2)$$

We will justify the a priori bound (2.1) in the end of Section 2.2. In the following, unless specified, it is denoted by

$$\int \cdot dx = \int_0^1 \cdot dx, \quad \int \cdot dt = \int_0^T \cdot dt \text{ for } T > 0.$$

2.1 Basic Energy Estimates

In this section, we start by some basic estimates under the a priori assumption (2.1). Moreover, the estimates in this section are independent of time. For a constant $C_0 > 0$, determined in (2.7), denote

$$E_0 = \mathcal{E}_0, \quad E_1 = \mathcal{E}_1 + C_0(\alpha^{6\gamma} + \beta^2)\mathcal{E}_0, \quad E_2 = \mathcal{E}_2. \quad (2.3)$$

The first lemma is concerning the kinetic and potential energy of (1.7).

Lemma 1 (Basic Energy Estimate) *For a smooth solution to (1.7), the kinetic and potential energies satisfy the following identity.*

$$\begin{aligned} & \frac{1}{2} \int x^2 \rho_0 v^2 dx + \frac{1}{\gamma - 1} \int r^2 r_x \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma dx \\ & + 2\mu \int \int \left(r^2 \frac{v_x^2}{r_x} + 2r_x v^2 \right) dx dt + \lambda \int \int r^2 r_x \left(\frac{v_x}{r_x} + 2\frac{v}{r} \right)^2 dx dt = E_0. \end{aligned} \quad (2.4)$$

Proof Multiply (1.7) with r^2v and integrate the resulting in spatial variable.

$$\int x^2 \rho_0 v_t v \, dx + \int \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_x r^2 v \, dx = \int \mathfrak{B}_x r^2 v \, dx + \int 4\mu \left(\frac{v}{r} \right)_x r^2 v \, dx. \quad (2.5)$$

Integration by parts yields

$$\begin{aligned} & \int \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_x r^2 v \, dx - \int \mathfrak{B}_x r^2 v \, dx = - \int \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma (r^2 v)_x \, dx \\ & + \int \mathfrak{B} (r^2 v)_x \, dx = \frac{d}{dt} \left\{ \frac{1}{\gamma - 1} \int r^2 r_x \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \, dx \right\} + \int \mathfrak{B} (r^2 v)_x \, dx \end{aligned}$$

Moreover,

$$\begin{aligned} & \int \mathfrak{B} (r^2 v)_x \, dx - \int 4\mu \left(\frac{v}{r} \right)_x r^2 v \, dx \\ & = \int (2\mu + \lambda) r^2 \frac{v_x^2}{r_x} + 4\lambda r v v_x + (4\mu + 4\lambda) r_x v^2 \, dx \\ & = 2\mu \int \left(r^2 \frac{v_x^2}{r_x} + 2r_x v^2 \right) \, dx + \lambda \int r^2 r_x \left(\frac{v_x}{r_x} + 2\frac{v}{r} \right)^2 \, dx \end{aligned}$$

Therefore, (2.5) can be written as

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \int x^2 \rho_0 v^2 \, dx + \frac{1}{\gamma - 1} \int r^2 r_x \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \, dx \right\} \\ & + 2\mu \int \left(r^2 \frac{v_x^2}{r_x} + 2r_x v^2 \right) \, dx + \lambda \int r^2 r_x \left(\frac{v_x}{r_x} + 2\frac{v}{r} \right)^2 \, dx = 0 \end{aligned} \quad (2.6)$$

Integrating over temporal variable yield (2.4). \square

The next lemma concerns the time derivative of (1.7).

Lemma 2 *There is a constant $C_0 > 0$ depending on $\mu, \lambda, \bar{\rho}_0$ such that*

$$\begin{aligned} & \frac{1}{2} \int x^2 \rho_0 v_t^2 \, dx + \mu \int \int \left(r^2 \frac{v_{xt}^2}{r_x} + 2r_x v_t^2 \right) \, dx \\ & \leq \mathcal{E}_1 + C_0 (\alpha^{6\gamma} + \beta^2) \mathcal{E}_0 = E_1. \end{aligned} \quad (2.7)$$

Proof Multiply (1.7) with r^2 and take the time derivative of the resulting equation. It follows,

$$\begin{aligned} x^2 \rho_0 v_{tt} + r^2 \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_{xt} &= r^2 \mathfrak{B}_{xt} + 4\mu r^2 \left(\frac{v}{r} \right)_{xt} \\ &+ 2rv \left(\mathfrak{B}_x + 4\mu \left(\frac{v}{r} \right)_x \right) - 2rv \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_x. \end{aligned} \quad (2.8)$$

(2.8) would be complemented with additional boundary conditions

$$\left\{ \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_t - \mathfrak{B}_t \right\} \Big|_{x=1} = 0, \quad v_t(0, t) = 0. \quad (2.9)$$

Multiply (2.8) with v_t and integrate the resulting in spatial variable. After integration by parts, it follows,

$$\begin{aligned} &\int x^2 \rho_0 v_{tt} v_t dx - \int \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_t (r^2 v_t)_x dx + \int \mathfrak{B}_t (r^2 v_t)_x dx \\ &- \int 4\mu r^2 v_t \left(\frac{v}{r} \right)_{xt} dx = 2 \int \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma (rvv_t)_x dx - 2 \int \mathfrak{B} (rvv_t)_x dx \\ &+ 8\mu \int rvv_t \left(\frac{v}{r} \right)_x dx := L_1 + L_2 + L_3. \end{aligned} \quad (2.10)$$

Notice, by using (2.1)

$$\begin{aligned} &\left| \int \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_t (r^2 v_t)_x dx \right| = \left| \gamma \int \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \frac{(r^2 r_x)_t}{r^2 r_x} (r^2 v_t)_x dx \right| \\ &\leq C\alpha^{3\gamma} \int \left| \left(\frac{v_x}{r_x} + 2\frac{v}{r} \right) (r^2 v_t)_x \right| dx \\ &\leq C\alpha^{3\gamma} \int |4r_x vv_t| + |2rvv_{xt}| + |2rv_tv_x| + \left| \frac{r^2}{r_x} v_x v_{xt} \right| dx \\ &\leq \delta \left\{ \int r^2 \frac{v_{xt}^2}{r_x} dx + \int r_x v_t^2 dx \right\} + C_\delta \alpha^{6\gamma} \left\{ \int r_x v^2 dx + \int r^2 \frac{v_x^2}{r_x} dx \right\}. \end{aligned}$$

In the meantime,

$$\begin{aligned} &\int \mathfrak{B}_t (r^2 v_t)_x dx - \int 4\mu r^2 v_t \left(\frac{v}{r} \right)_{xt} dx \\ &= 2\mu \int \left(r^2 \frac{v_{xt}^2}{r_x} + 2r_x v_t^2 \right) dx + \lambda \int r^2 r_x \left(\frac{v_{xt}}{r_x} + 2\frac{v_t}{r} \right)^2 dx - L_4, \end{aligned}$$

where

$$\begin{aligned} L_4 = & (4\mu + 2\lambda) \int \left(\frac{v_x}{r_x} \right) r v_t v_x dx + (2\mu + \lambda) \int \left(\frac{v_x}{r_x} \right) \frac{r^2 v_x v_{xt}}{r_x} dx \\ & + (8\mu + 4\lambda) \int \left(\frac{v}{r} \right) r_x v v_t dx + 2\lambda \int \left(\frac{v}{r} \right) r v v_{xt} dx - 8\mu \int \left(\frac{v}{r} \right) r v_x v_t dx. \end{aligned}$$

Again, using (2.1) and applying Cauchy's inequality, it follows,

$$\begin{aligned} L_1, L_2, L_3, L_4 \leq & C(\alpha^{3\gamma} + \beta) \left\{ \int r v_t v_x dx + \int r_x v v_t dx \right. \\ & \left. + \int r v v_{xt} dx + \int \frac{r^2 v_x v_{xt}}{r_x} dx \right\} \leq \delta \left\{ \int r^2 \frac{v_{xt}^2}{r_x} dx + \int r_x v_t^2 dx \right\} \\ & + C_\delta(\alpha^{6\gamma} + \beta^2) \left\{ \int r^2 \frac{v_x^2}{r_x} dx + \int r_x v^2 dx \right\}. \end{aligned}$$

Summing up the inequality above, it follows from (2.10),

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \int x^2 \rho_0 v_t^2 dx \right\} + \int 2\mu \left(r^2 \frac{v_{xt}^2}{r_x} + 2r_x v_t^2 \right) + \lambda r^2 r_x \left(\frac{v_{xt}}{r_x} + 2\frac{v_t}{r} \right)^2 dx \\ & \leq 5\delta \left\{ \int \left(r^2 \frac{v_{xt}^2}{r_x} + 2r_x v_t^2 \right) dx \right\} + C_\delta(\alpha^{6\gamma} + \beta^2) \left\{ \int r^2 \frac{v_x^2}{r_x} dx + \int r_x v^2 dx \right\} \end{aligned} \quad (2.11)$$

We shall choose δ small enough. Then (2.7) follows from integration in temporal variable of (2.11) together with (2.4). \square

2.2 Uniform Estimates

The aim in this section is to show some uniform (in time) estimates. With these estimates, it would be able to design the restriction on initial data, with which under the a prior assumption (2.1), it can be shown,

$$\frac{x^2}{r^2 r_x} < \alpha^3, \left| \frac{v_x}{r_x} \right|, \left| \frac{v}{r} \right| < \beta \quad (2.12)$$

for a smooth solution to (1.7). In particular, the a prior bounds in (2.1) can be justified. Moreover, the point-wise bounds of

$$\frac{x^2}{r^2 r_x}, \frac{v_x}{r_x}, \frac{v}{r}$$

are independent of time, which would be important ingredients to establish the propagation of regularity in the next section. In this and the following sections, we agree that the constant $C > 0$ is a universal constant which might be different from line to line and depends only on $\bar{\rho}_0, \mu, \lambda, \gamma$.

A direct consequence of (2.2) is $R(t) = r(1, t) \geq \alpha^{-1}$. With such a prior lower bound of the radius, the value of v on the boundary ($x = 1$) admits the following estimates.

Lemma 3 *Under the same assumptions as in Lemma 1, there is a constant $C > 0$ such that,*

$$\int v^2|_{x=1} dt \leq C\alpha E_0, \quad (2.13)$$

$$\int v_t^2|_{x=1} dt \leq C\alpha E_1, \quad (2.14)$$

$$v^2|_{x=1} \leq C\alpha E_0 + C\alpha E_1. \quad (2.15)$$

Proof As we have mentioned, $\exists 0 < \sigma < \alpha^{-1}/2$, satisfying $R(t) - \sigma > \sigma$. For instant, we shall take $\sigma = \alpha^{-1}/4$. Then we have, by applying the mean value theorem, fundamental theorem of calculus and Cauchy's inequality,

$$\begin{aligned} v^2|_{x=1} &= u^2(R(t), t) \leq \sigma^{-1} \int_{R(t)-\sigma}^{R(t)} u^2 dr + \int_{R(t)-\sigma}^{R(t)} (u^2)_r dr \\ &\leq C\sigma^{-1} \int_{R(t)-\sigma}^{R(t)} u^2 dr + C\sigma \int_{R(t)-\sigma}^{R(t)} u_r^2 dr \leq C\sigma^{-1} \int_{R(t)-\sigma}^{R(t)} u^2 dr \\ &\quad + C\sigma(R(t) - \sigma)^{-2} \int_{R(t)-\sigma}^{R(t)} r^2 u_r^2 dr. \end{aligned}$$

Therefore,

$$\begin{aligned} v^2|_{x=1} &\leq C\sigma^{-1} \left(\int_0^{R(t)} u^2 dr + \int_0^{R(t)} r^2 u_r^2 dr \right) \\ &= C\alpha \left(\int r_x v^2 dx + \int r^2 \frac{v_x^2}{r_x} dx \right), \end{aligned} \quad (2.16)$$

where we have used the fact $(R(t) - \sigma)^{-2} < \sigma^{-2}$. Then from (2.4) it holds

$$\int v^2|_{x=1} dt \leq C\alpha E_0. \quad (2.17)$$

Meanwhile, by (2.7), similar argument as in (2.16) yields

$$\int v_t^2|_{x=1} dt \leq C\alpha \int \left(\int r_x v_t^2 dx + \int r^2 \frac{v_{xt}^2}{r_x} dx \right) dt \leq C\alpha E_1. \quad (2.18)$$

Consequently,

$$v^2|_{x=1} \leq C \int (v^2|_{x=1} + v_t^2|_{x=1}) dt \leq C\alpha E_0 + C\alpha E_1. \quad (2.19)$$

□

The following lemma is the key ingredient in this work. It shows the time-integrability of

$$\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma}, \left| \frac{v_x}{r_x} \right|^2, \left| \frac{v}{r} \right|^2.$$

Lemma 4 *Under the same assumptions as in Lemma 1, there exists a constant $C > 0$ such that*

$$\begin{aligned} & \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right\|_{L_x^\infty} + C \int \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} dt + C \int \left\| \frac{v_x}{r_x} + 2 \frac{v}{r} \right\|_{L_x^\infty}^2 dt \\ & \leq \bar{\rho}^\gamma + C\alpha^5 E_1 + C\alpha^3 E_0, \end{aligned} \quad (2.20)$$

$$\int \left\| \frac{v_x}{r_x} - \frac{v}{r} \right\|_{L_x^\infty}^2 dt \leq C\bar{\rho}^\gamma + C\alpha^5 E_1 + C\alpha^3 E_0, \quad (2.21)$$

$$\int \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2 + \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 dt \leq C\bar{\rho}^\gamma + C\alpha^5 E_1 + C\alpha^3 E_0. \quad (2.22)$$

Proof Integrate (1.7) over $(x, 1)$ in spatial variable for $0 < x < 1$. It follows,

$$-(2\mu + \lambda) \frac{(r^2 v)_x}{r^2 r_x} + \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma = \int_x^1 \left(\frac{x}{r} \right)^2 \rho_0 v_t dx - 4\mu \frac{v}{r} \Big|_{x=1}. \quad (2.23)$$

Notice

$$(r^2 v)_x = \frac{1}{3} (\partial_t r^3)_x = (r^2 r_x)_t.$$

Taking square on both sides of (2.23) yields,

$$\begin{aligned}
& -2(2\mu + \lambda) \frac{(r^2 r_x)_t}{r^2 r_x} \cdot \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma + \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} + \left((2\mu + \lambda) \frac{(r^2 r_x)_t}{r^2 r_x} \right)^2 \\
& \leq \left(\int_x^1 \left(\frac{x}{r} \right)^2 \rho_0 v_t dx \right)^2 + 8\mu^2 \left(\frac{v}{r} \Big|_{x=1} \right)^2 \\
& \leq \int_x^1 \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx \cdot \int_x^1 \left(\frac{x}{r} \right)^2 \rho_0 dx + C \frac{v^2}{r^2} \Big|_{x=1}.
\end{aligned} \tag{2.24}$$

Notice, from (2.1), (2.2) and (1.10)

$$\begin{aligned}
& -\frac{(r^2 r_x)_t}{r^2 r_x} \cdot \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma = \frac{1}{\gamma} \frac{d}{dt} \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma, \\
& \int_x^1 \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx \cdot \int_x^1 \left(\frac{x}{r} \right)^2 \rho_0 dx \leq C\alpha^5 \int r_x v_t^2 dx.
\end{aligned}$$

Integration over $(0, t)$ in temporal variable of (2.24) then implies

$$\begin{aligned}
& \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma + C \int \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} dt + C \int \left(\frac{(r^2 r_x)_t}{r^2 r_x} \right)^2 dt \\
& \leq \rho_0^\gamma + C\alpha^5 \int dt \int r_x v_t^2 dx + C \int \frac{v^2}{r^2} \Big|_{x=1} dt \\
& \leq \bar{\rho}^\gamma + C\alpha^5 E_1 + C\alpha^3 E_0.
\end{aligned} \tag{2.25}$$

where the last inequality follows from (2.2), (2.7) and (2.13). (2.20) follows by noticing

$$\frac{(r^2 r_x)_t}{r^2 r_x} = \frac{v_x}{r_x} + 2\frac{v}{r}.$$

On the other hand, multiply (1.7) with r^3 and integrate the resulting over $(0, x)$ in spatial variable for $0 < x < 1$. After integration by parts, it holds,

$$\begin{aligned}
& \int_0^x x^2 r \rho_0 v_t dx + \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma r^3 - \int_0^x \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma (r^3)_x dx \\
& = (2\mu + \lambda) \left(\frac{(r^2 v)_x}{r^2 r_x} r^3 - \int_0^x \frac{(r^2 v)_x}{r^2 r_x} (r^3)_x dx \right).
\end{aligned} \tag{2.26}$$

Direct calculation yields,

$$\frac{(r^2 v)_x}{r^2 r_x} r^3 - \int_0^x \frac{(r^2 v)_x}{r^2 r_x} (r^3)_x dx = \left(\frac{v_x}{r_x} - \frac{v}{r} \right) \cdot r^3.$$

Moreover, similar as before, from (1.10) and (2.2)

$$\begin{aligned}
\int_0^x x^2 r \rho_0 v_t dx &\leq \left(\int \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx \right)^{1/2} \left(\int_0^x \rho_0 x^2 r^4 dx \right)^{1/2} \\
&\leq C \left(\int \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx \right)^{1/2} \left(x \frac{x^2}{r^2} r^6 \right)^{1/2} \\
&\leq C \alpha \left(\int \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx \right)^{1/2} r^3, \\
\int_0^x \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma (r^3)_x dx &\leq \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right\|_{L_x^\infty} \int_0^x (r^3)_x dx \leq \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right\|_{L_x^\infty} r^3.
\end{aligned}$$

Therefore, (2.26) implies

$$\left| \frac{v_x}{r_x} - \frac{v}{r} \right| \cdot r^3 \leq C \alpha \left(\int \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx \right)^{1/2} r^3 + C \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right\|_{L_x^\infty} r^3,$$

or

$$\begin{aligned}
\left| \frac{v_x}{r_x} - \frac{v}{r} \right|^2 &\leq C \alpha^2 \int \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + C \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} \\
&\leq C \alpha^5 \int r_x v_t^2 dx + C \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty},
\end{aligned} \tag{2.27}$$

where (1.10) and (2.1) is applied in the last inequality. Integrating the above inequality over temporal variable, together with (2.7) and (2.20) then yields (2.21). (2.22) follows from (2.20) and (2.21). \square

We shall need the following lemmas concerning the interior energy inspired by [23], which will claim extra integrability of the unknowns.

Lemma 5 *Under the same assumptions as in Lemma 1, there exists a constant $C > 0$ such that,*

$$\int \frac{v_x^2}{r_x} dx + \int r_x \left(\frac{v}{r} \right)^2 dx \leq C \alpha^2 E_0 + C(\alpha^2 + \Gamma) E_1 + C \alpha^2 \Gamma^{2\gamma-1}, \tag{2.28}$$

where

$$\Gamma =: \min \{ \bar{\rho} \alpha^3, (\bar{\rho}^\gamma + C \alpha^5 E_1 + C \alpha^3 E_0)^{1/\gamma} \}. \tag{2.29}$$

As a consequence of (2.1), (1.10) and (2.20),

$$\max_{0 \leq x \leq 1} \frac{x^2 \rho_0}{r^2 r_x} \leq \Gamma. \quad (2.30)$$

Proof Multiply (1.7) with v and integrate the resulting in spatial variable. Then it holds

$$\int \left(\frac{x}{r}\right)^2 \rho_0 v_t v dx - \int \left(\frac{x^2 \rho_0}{r^2 r_x}\right)^\gamma v_x dx = -(2\mu + \lambda) \int \frac{(r^2 v)_x}{r^2 r_x} v_x dx + 4\mu \frac{v^2}{r} \Big|_{x=1}. \quad (2.31)$$

where the integration by parts is applied with the boundary condition (1.8). Notice, integration by parts again yields,

$$\begin{aligned} (2\mu + \lambda) \int \frac{(r^2 v)_x}{r^2 r_x} v_x dx &= (2\mu + \lambda) \int \frac{v_x^2}{r_x} dx + (2\mu + \lambda) \int \frac{2v v_x}{r} dx \\ &= (2\mu + \lambda) \int \frac{v_x^2}{r_x} dx + (2\mu + \lambda) \int r_x \frac{v^2}{r^2} dx + (2\mu + \lambda) \frac{v^2}{r} \Big|_{x=1} \end{aligned}$$

Moreover, from (1.10), (2.2), (2.30) and Cauchy's inequality,

$$\begin{aligned} \int \left(\frac{x}{r}\right)^2 \rho_0 v_t v dx &\leq \delta \int r_x \left(\frac{v}{r}\right)^2 dx + C_\delta \int \frac{x^2 \rho_0}{r^2 r_x} x^2 \rho_0 v_t^2 dx \\ &\leq \delta \int r_x \left(\frac{v}{r}\right)^2 dx + C_\delta \Gamma \int x^2 \rho_0 v_t^2 dx \\ \int \left(\frac{x^2 \rho_0}{r^2 r_x}\right)^\gamma v_x dx &\leq \delta \int \frac{v_x^2}{r_x} dx + C_\delta \int \frac{x^2 \rho_0}{r^2} \left(\frac{x^2 \rho_0}{r^2 r_x}\right)^{2\gamma-1} dx \\ &\leq \delta \int \frac{v_x^2}{r_x} dx + C_\delta \alpha^2 \Gamma^{2\gamma-1}. \end{aligned}$$

Therefore, from (2.31) it follows

$$\begin{aligned} (2\mu + \lambda) \left(\int \frac{v_x^2}{r_x} dx + \int r_x \left(\frac{v}{r}\right)^2 dx \right) &\leq 2\delta \left(\int r_x \left(\frac{v}{r}\right)^2 dx + \int \frac{v_x^2}{r_x} dx \right) \\ &\quad + C_\delta \Gamma \int x^2 \rho_0 v_t^2 dx + C_\delta \alpha^2 \Gamma^{2\gamma-1} + C \frac{v^2}{r} \Big|_{x=1} \\ &\leq 2\delta \left(\int r_x \left(\frac{v}{r}\right)^2 dx + \int \frac{v_x^2}{r_x} dx \right) + C_\delta \Gamma E_1 + C_\delta \alpha^2 \Gamma^{2\gamma-1} \\ &\quad + C \alpha^2 E_0 + C \alpha^2 E_1, \end{aligned} \quad (2.32)$$

where the last inequality follows from (2.2), (2.7), (2.15). By choosing δ small enough, (2.28) follows. \square

Lemma 6 *Under the same assumptions as in Lemma 1, there exists a constant $C > 0$ such that,*

$$\begin{aligned} & \frac{1}{2} \int \left(\frac{x}{r}\right)^2 \rho_0 v_t^2 dx + \frac{2\mu + \lambda}{2} \int \int \left(r_x \frac{v_t^2}{r^2} + \frac{v_{xt}^2}{r_x}\right) dx dt \\ & \leq E_2 + C(\alpha^2 E_0 + (\alpha^2 + \Gamma)E_1 + \alpha^2 \Gamma^{2\gamma-1}) \times (\alpha^3 E_0 + \alpha^5 E_1 + 1). \end{aligned} \quad (2.33)$$

Proof Taking the time derivative of (1.7),

$$\left(\frac{x}{r}\right)^2 \rho_0 v_{tt} + \left[\left(\frac{x^2 \rho_0}{r^2 r_x}\right)^\gamma\right]_{xt} = (2\mu + \lambda) \left[\frac{(r^2 v)_x}{r^2 r_x}\right]_{xt} + 2 \frac{x^2 v}{r^2 r} \rho_0 v_t. \quad (2.34)$$

Multiply this equation with v_t and integrate the resulting in spatial variable.

$$\begin{aligned} & \int \left(\frac{x}{r}\right)^2 \rho_0 v_{tt} v_t dx - \int v_{xt} \left[\left(\frac{x^2 \rho_0}{r^2 r_x}\right)^\gamma\right]_t dx \\ & = -(2\mu + \lambda) \int v_{xt} \left[\frac{(r^2 v)_x}{r^2 r_x}\right]_t dx + 4\mu v_t \left(\frac{v}{r}\right)_t \Big|_{x=1} + 2 \int \frac{x^2 v}{r^2 r} \rho_0 v_t^2 dx \end{aligned} \quad (2.35)$$

Direct calculation yields,

$$\begin{aligned} & \frac{d}{dt} \left\{ \frac{1}{2} \int \left(\frac{x}{r}\right)^2 \rho_0 v_t^2 dx \right\} + (2\mu + \lambda) \int \left(r_x \frac{v_t^2}{r^2} + \frac{v_{xt}^2}{r_x}\right) dx \\ & = \int \frac{x^2 \rho_0}{r^2} \frac{v}{r} v_t^2 dx + (2\mu + \lambda) \int \left(2 \frac{v^2}{r^2} + \frac{v_x^2}{r_x^2}\right) v_{xt} dx \\ & \quad - \gamma \int \left(\frac{x^2 \rho_0}{r^2 r_x}\right)^\gamma \left(\frac{v_x v_{xt}}{r_x} + 2 \frac{v v_{xt}}{r}\right) dx \\ & \quad + \left[-(2\mu + \lambda) \frac{v_t^2}{r} + 4\mu v_t \left(\frac{v}{r}\right)_t \right] \Big|_{x=1} := I_1 + I_2 + I_3 + I_4. \end{aligned} \quad (2.36)$$

We estimate I_1, \dots, I_4 as follows. By noticing (2.2), (2.15) and (2.28),

Cauchy's inequality yields

$$\begin{aligned}
I_2 &\leq \delta \int \frac{v_{xt}^2}{r_x} dx + C_\delta \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2 \int \frac{v_x^2}{r_x} dx + C_\delta \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 \int r_x \frac{v^2}{r^2} dx \\
&\leq \delta \int \frac{v_{xt}^2}{r_x} dx + C_\delta (\alpha^2 E_0 + (\alpha^2 + \Gamma) E_1 + \alpha^2 \Gamma^{2\gamma-1}) \\
&\quad \times \left\{ \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2 + \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 \right\} \\
I_3 &\leq \delta \int \frac{v_{xt}^2}{r_x} dx + C_\delta \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} \cdot \left\{ \int \frac{v_x^2}{r_x} dx + \int r_x \frac{v^2}{r^2} dx \right\} \\
&\leq \delta \int \frac{v_{xt}^2}{r_x} dx + C_\delta (\alpha^2 E_0 + (\alpha^2 + \Gamma) E_1 + \alpha^2 \Gamma^{2\gamma-1}) \cdot \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} \\
I_4 &\leq C (\alpha v_t^2 + \alpha^3 v^4)|_{x=1} \leq C \alpha v_t^2|_{x=1} + C \alpha^4 (E_0 + E_1) v^2|_{x=1}
\end{aligned}$$

Moreover, after plugging (1.7) into I_1 , applying integration by parts to the resulting expression then yields, together with (2.2),

$$\begin{aligned}
I_1 &= \int \frac{v}{r} v_t \left((2\mu + \lambda) \left(\frac{(r^2 v)_x}{r^2 r_x} \right)_x - \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_x \right) dx \\
&= \int \left(\frac{v}{r} v_t \right)_x \left(\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma - (2\mu + \lambda) \frac{(r^2 v)_x}{r^2 r_x} \right) dx + 4\mu \frac{v^2}{r^2} v_t \Big|_{x=1} \\
&\leq \delta \left\{ \int r_x \frac{v_t^2}{r^2} dx + \int \frac{v_{xt}^2}{r_x} dx \right\} + C_\delta \left\{ \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} \right. \\
&\quad \left. + \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2 + \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 \right\} \cdot \left\{ \int \frac{v_x^2}{r_x} dx + \int r_x \frac{v^2}{r^2} dx \right\} + C (\alpha v_t^2 + \alpha^3 v^4)|_{x=1}.
\end{aligned}$$

Therefore, similarly as consequences of (2.15) and (2.28),

$$\begin{aligned}
I_1 &\leq \delta \left\{ \int r_x \frac{v_t^2}{r^2} dx + \int \frac{v_{xt}^2}{r_x} dx \right\} + C_\delta (\alpha^2 E_0 + (\alpha^2 + \Gamma) E_1 + \alpha^2 \Gamma^{2\gamma-1}) \\
&\quad \times \left\{ \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} + \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2 + \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 \right\} \\
&\quad + C \alpha v_t^2|_{x=1} + C \alpha^4 (E_0 + E_1) v^2|_{x=1}
\end{aligned}$$

Therefore by choosing δ small enough, integration of (2.36) in temporal variable yields,

$$\begin{aligned}
& \frac{1}{2} \int \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + \frac{2\mu + \lambda}{2} \int \int \left(r_x \frac{v_t^2}{r^2} + \frac{v_{xt}^2}{r_x} \right) dx dt \\
& \leq \frac{1}{2} \int \rho_0 u_1^2 dx + C(\alpha^2 E_0 + (\alpha^2 + \Gamma) E_1 + \alpha^2 \Gamma^{2\gamma-1}) \\
& \quad \times \int \left\{ \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} + \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2 + \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 \right\} dt \\
& \quad + C\alpha \int v_t^2|_{x=1} dt + C\alpha^4(E_0 + E_1) \int v^2|_{x=1} dt \\
& \leq C\alpha^5 E_0(E_0 + E_1) + C\alpha^2 E_1 + E_2 \\
& \quad + C(\alpha^2 E_0 + (\alpha^2 + \Gamma) E_1 + \alpha^2 \Gamma^{2\gamma-1}) \times (\alpha^3 E_0 + \alpha^5 E_1 + \bar{\rho}^\gamma)
\end{aligned} \tag{2.37}$$

where the last inequality is from (2.13), (2.14), (2.20) and (2.22). Then (2.33) follows by noticing $\alpha > 1$. \square

Now we have enough materials to show the point-wise bounds of

$$\frac{x^2}{r^2 r_x}, \frac{v_x}{r_x}, \frac{v}{r},$$

under the a prior assumption (2.1). In particular, the bounds are independent of time. More precisely,

Lemma 7 *Under the same assumptions as in Lemma 1, there exist $C > 0$ such that,*

$$\begin{aligned}
& \left\| \frac{v_x}{r_x} + 2\frac{v}{r} \right\|_{L_x^\infty}^2 \leq C\alpha^2((\alpha^2 E_0 + (\alpha^2 + \Gamma) E_1 + \alpha^2 \Gamma^{2\gamma-1}) \\
& \quad \times (\alpha^3 E_0 + \alpha^5 E_1 + 1) + E_2) + \Gamma^{2\gamma},
\end{aligned} \tag{2.38}$$

$$\begin{aligned}
& \left\| \frac{v_x}{r_x} - \frac{v}{r} \right\|_{L_x^\infty}^2 \leq C\alpha^2((\alpha^2 E_0 + (\alpha^2 + \Gamma) E_1 + \alpha^2 \Gamma^{2\gamma-1}) \\
& \quad \times (\alpha^3 E_0 + \alpha^5 E_1 + 1) + E_2) + \Gamma^{2\gamma},
\end{aligned} \tag{2.39}$$

$$\left\| \frac{x^2}{r^2 r_x} \right\|_{L_x^\infty} \leq \exp \left\{ C\alpha^{5/2}((1 + \beta) E_0 + E_1)^{1/2} \right. \tag{2.40}$$

$$\left. + C\alpha^{5/2}(\alpha^3 E_0 + \alpha^5 E_1 + 1)^{1/2} E_1^{1/2} \right\}. \tag{2.41}$$

Proof From (2.24), (1.10), (2.2),

$$\begin{aligned}
(2\mu + \lambda)^2 \left(\frac{(r^2 r_x)_t}{r^2 r_x} \right)^2 &\leq \int_x^1 \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx \cdot \int_x^1 \left(\frac{x}{r} \right)^2 \rho_0 dx + C \frac{v^2}{r^2} \Big|_{x=1} \\
&\quad - \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} + 2(2\mu + \lambda) \frac{(r^2 r_x)_t}{r^2 r_x} \cdot \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \\
&\leq \delta \left(\frac{(r^2 r_x)_t}{r^2 r_x} \right)^2 + C\alpha^2 \int \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + C\alpha^2 v^2 \Big|_{x=1} + C_\delta \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma}.
\end{aligned} \tag{2.42}$$

After choosing δ small enough, (1.10), (2.1), (2.15), (2.30) and (2.33) implies

$$\begin{aligned}
\left(\frac{(r^2 r_x)_t}{r^2 r_x} \right)^2 &\leq C\alpha^2 ((\alpha^2 E_0 + (\alpha^2 + \Gamma)E_1 + \alpha^2 \Gamma^{2\gamma-1}) \times (\alpha^3 E_0 + \alpha^5 E_1 + 1) \\
&\quad + E_2) + C\alpha^3 (E_0 + E_1) + \Gamma^{2\gamma}.
\end{aligned} \tag{2.43}$$

Then (2.38) follows by noticing $\alpha > 1$ and the fact

$$\frac{(r^2 r_x)_t}{r^2 r_x} = \frac{v_x}{r_x} + 2\frac{v}{r}.$$

To show (2.39), from (2.27), (2.33), (2.30)

$$\begin{aligned}
\left| \frac{v_x}{r_x} - \frac{v}{r} \right|^2 &\leq C\alpha^2 \int \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + C \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} \\
&\leq C\alpha^2 ((\alpha^2 E_0 + (\alpha^2 + \Gamma)E_1 + \alpha^2 \Gamma^{2\gamma-1}) \times (\alpha^3 E_0 + \alpha^5 E_1 + 1) \\
&\quad + E_2) + C\Gamma^{2\gamma}.
\end{aligned} \tag{2.44}$$

In order to show (2.41), from (2.23)

$$\begin{aligned}
-(2\mu + \lambda) \frac{d}{dt} \ln r^2 r_x &\leq -(2\mu + \lambda) \frac{(r^2 r_x)_t}{r^2 r_x} + \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma = \int_x^1 \frac{x^2}{r^2} \rho_0 v_t dx \\
-4\mu \frac{d}{dt} \ln R(t) &= \frac{d}{dt} \left\{ \int_x^1 \frac{x^2}{r^2} \rho_0 v dx - 4\mu \ln R(t) \right\} + 2 \int_x^1 \frac{x^2}{r^2} \frac{v}{r} \rho_0 v dx,
\end{aligned}$$

where

$$R(t) = r(x = 1, t).$$

Integration in temporal variable yields, noticing $R(0) = 1$ and $\frac{r^2 r_x}{x^2}(x, t = 0) = 1$,

$$\begin{aligned} - (2\mu + \lambda) \ln \frac{r^2 r_x}{x^2} &\leq -4\mu \ln R(t) + \int_x^1 \frac{x^2}{r^2} \rho_0 v \, dx - \int_x^1 \rho_0 u_0 \, dx \\ &+ 2 \int \int_x^1 \frac{x^2}{r^2} \frac{v}{r} \rho_0 v \, dx \, dt. \end{aligned} \quad (2.45)$$

Denote

$$h = \int_x^1 \frac{x^2}{r^2} \rho_0 v \, dx - \int_x^1 \rho_0 u_0 \, dx + 2 \int \int_x^1 \frac{x^2}{r^2} \frac{v}{r} \rho_0 v \, dx \, dt.$$

Then, (2.45) can be written as

$$\frac{x^2}{r^2 r_x} \leq R(t)^{-\frac{4\mu}{2\mu+\lambda}} \exp \frac{h}{2\mu + \lambda}. \quad (2.46)$$

Denote

$$\mathcal{X} = \max_{0 \leq x \leq 1} \frac{x^2}{r^2 r_x}.$$

Then

$$R^3(t) = 3 \int_0^1 r^2 r_x \, dx \geq 3\mathcal{X}^{-1} \int_0^1 x^2 \, dx = \mathcal{X}^{-1}.$$

Therefore, from (2.46), it holds

$$\mathcal{X}^{\frac{2\mu+3\lambda}{3(2\mu+\lambda)}} \leq \exp(Ch). \quad (2.47)$$

Meanwhile, by applying Hölder inequality, (1.10)

$$\begin{aligned} h &\leq C \left(\int \frac{x^2}{r^2} \, dx \right)^{1/2} \left(\int \frac{x^2}{r^2} v^2 \, dx \right)^{1/2} - \int_x^1 \rho_0 u_0 \, dx \\ &+ C \left(\int \int \frac{x^2}{r^2} v^2 \, dx \, dt \right)^{1/2} \left(\int \int \frac{x^2}{r^2} \left(\frac{v}{r} \right)^2 \, dx \, dt \right)^{1/2} \end{aligned} \quad (2.48)$$

In addition, from (1.10), (2.1), (2.4), (2.7),

$$\begin{aligned}
\int \frac{x^2}{r^2} v^2 dx &\leq C\alpha^3 \int r_x v^2 dx \leq C\alpha^3 \int dt \int r_x v^2 dx \\
&+ C\alpha^3 \int dt \int (r_x v^2)_t dx \leq C\alpha^3 \left(1 + \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}\right) \int dt \int r_x v^2 dx \\
&+ C\alpha^3 \int dt \int r_x v_t^2 dx \leq C\alpha^3 ((1 + \beta)E_0 + E_1), \\
\left| \int_x^1 \rho_0 u_0 dx \right| &\leq C \left(\int u_0^2 dx \right)^{1/2}.
\end{aligned}$$

Notice, since $r_x(x, 0) = 1$,

$$\begin{aligned}
\int u_0^2 dx &\leq \sup_{t \geq 0} \int r_x v^2 dx \\
&\leq \int dt \int r_x v^2 dx + \int dt \int (r_x v^2)_t dx \leq C((1 + \beta)E_0 + E_1).
\end{aligned}$$

Also, (2.1), (2.2), (2.22), (2.4)

$$\begin{aligned}
\int \int \frac{x^2}{r^2} \left(\frac{v}{r} \right)^2 dx dt &\leq C\alpha^2 \int \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 dt \leq C\alpha^2 (\alpha^3 E_0 + \alpha^5 E_1 + 1), \\
\int \int \frac{x^2}{r^2} v^2 dx dt &\leq C\alpha^3 \int \int r_x v^2 dx \leq C\alpha^3 E_0.
\end{aligned}$$

Therefore, from (2.2), (2.48) and the fact $\alpha > 1$,

$$h \leq C\alpha^{5/2}((1 + \beta)E_0 + E_1)^{1/2} + C\alpha^{5/2}(\alpha^3 E_0 + \alpha^5 E_1 + 1)^{1/2} E_0^{1/2}. \quad (2.49)$$

Consequently,

$$\mathcal{X} \leq \exp \left\{ C\alpha^{5/2}((1 + \beta)E_0 + E_1)^{1/2} + C\alpha^{5/2}(\alpha^3 E_0 + \alpha^5 E_1 + 1)^{1/2} E_1^{1/2} \right\}. \quad (2.50)$$

□

With Lemma 7, it is possible to design the condition on $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2$ such that the a priori assumption (2.1) can be verified for a smooth solution.

Lemma 8 *For a fixed α , there is a constant $\epsilon_0 > 0$, depending on $\alpha, \bar{\rho}_0, \mu, \lambda, M$, such that for $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \leq \epsilon_0$, $\exists \beta_0 = \beta_0(\alpha, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, M)$, $\beta_1 = \beta_1(\alpha, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, M)$, $\beta_1 \geq \beta_0 > M$, satisfying the following. If $\beta_0 \leq \beta \leq \beta_1$ in (2.1), then the following inequality holds,*

$$\left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2, \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 \leq \frac{1}{2} \beta^2 < \beta^2. \quad (2.51)$$

In addition, there exists a constant $\epsilon_1 \leq \epsilon_0$, depending on $\alpha, \bar{\rho}, \mu, \lambda, M$, such that for $\mathcal{E}_0, \mathcal{E}_1 \leq \epsilon_1, \mathcal{E}_2 \leq \epsilon_0$, if $\beta \leq \beta_1(\alpha, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, M)$, it holds,

$$\left\| \frac{x^2}{r^2 r_x} \right\|_{L_x^\infty} \leq \alpha^{3/2} < \alpha^3. \quad (2.52)$$

Moreover, ϵ_0, ϵ_1 can be chosen such that,

$$\epsilon_0 = \epsilon_1 = \epsilon(\alpha), \quad (2.53)$$

with $\epsilon(\alpha)$ being a bounded continuous function of $\alpha \in (1, +\infty)$. We denote the following quantity which is the constant in (1.17),

$$\bar{\epsilon} := \sup_{1 < \alpha < +\infty} \epsilon(\alpha) < \infty. \quad (2.54)$$

Proof From (2.38), (2.39), there exists $C_1 > 0$ such that

$$\begin{aligned} \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2, \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 &\leq C_1 \alpha^2 ((\alpha^2 E_0 + (\alpha^2 + \Gamma) E_1 + \alpha^2 \Gamma^{2\gamma-1}) \\ &\quad \times (\alpha^3 E_0 + \alpha^5 E_1 + 1) + E_2) + C_1 \Gamma^{2\gamma}. \end{aligned} \quad (2.55)$$

From definition of E_0, E_1, E_2 in (2.3) and (2.29), (2.55) can be written as,

$$\left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2, \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 \leq V_\alpha(\beta, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2), \quad (2.56)$$

for some positive, increasing, continuous function V_α satisfying,

$$V_\alpha(0, 0, 0, 0) = C_1 \alpha^4 \Gamma_0^{2\gamma-1} + C_1 \Gamma_0^{2\gamma}, \quad \Gamma_0 = \min \{ \bar{\rho}_0 \alpha^3, \bar{\rho}_0 \}.$$

In addition, there exist $C, c > 0$ and positive integers l, n such that,

$$V_\alpha \leq C \alpha^l + A_1 \beta + A_2 \beta^n, \quad (2.57)$$

for some non-negative, increasing, continuous functions $A_1 = A_1(\alpha, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$, $A_2 = A_2(\alpha, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2)$ satisfying

$$A_1(\alpha, 0, 0, 0) = 0, \quad A_2(\alpha, 0, 0, 0) = 0.$$

Moreover, $A_1, A_2 > 0$ for $(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) \neq (0, 0, 0)$. Therefore, for some $\epsilon_0 > 0$ depending on $\alpha, \bar{\rho}_0, \mu, \lambda, M$, it holds for $0 < \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \leq \epsilon_0$, the set

$$S_\alpha := \left\{ \beta \geq M \mid V_\alpha(\beta, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) \leq \frac{1}{2}\beta^2 \right\} \quad (2.58)$$

is nonempty and bounded. Define

$$\beta_0 = \beta_0(\alpha, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, M) := \inf S_\alpha, \quad \beta_1 = \beta_1(\alpha, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, M) = \sup S_\alpha. \quad (2.59)$$

In this notation, $S_\alpha = [\beta_0, \beta_1]$. Then for $\beta \in S_\alpha$, if $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 \leq \epsilon_0$,

$$\left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2, \quad \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 \leq V_\alpha(\beta, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2) \leq \frac{1}{2}\beta^2 < \beta^2.$$

To show (2.52), from (2.41) we have,

$$\left\| \frac{x^2}{r^2 r_x} \right\|_{L_x^\infty} \leq \exp \{ W_\alpha(\beta, \mathcal{E}_0, \mathcal{E}_1) \}. \quad (2.60)$$

W_α is a non-negative, increasing, continuous function such that

$$W_\alpha \leq A_3 \alpha^l, \quad (2.61)$$

for some $l > 0$, where $A_3 = A_3(\beta, \mathcal{E}_0, \mathcal{E}_1)$ is a non-negative, increasing, continuous function and

$$A_3(\beta, 0, 0) = 0.$$

Therefore, there exists $\epsilon_1 \leq \epsilon_0$ such that for $\mathcal{E}_0, \mathcal{E}_1 \leq \epsilon_1, \mathcal{E}_2 \leq \epsilon_0, \beta \leq \beta_1$,

$$W_\alpha(\beta, \mathcal{E}_0, \mathcal{E}_1) \leq W_\alpha(\beta_1, \mathcal{E}_0, \mathcal{E}_1) \leq \frac{3}{2} \ln \alpha,$$

and therefore, since $\alpha > 1$,

$$\left\| \frac{x^2}{r^2 r_x} \right\|_{L_x^\infty} \leq \alpha^{3/2} < \alpha^3. \quad (2.62)$$

(2.53), (2.54) follow from (2.57), (2.61) and the continuity of V_α, W_α in $\alpha \in (1, +\infty)$ and similar arguments. \square

2.3 Regularity

In this section, the goal is to study the regularity propagated by the strong solution to (1.7) with the a prior bound (2.1). It is further assumed

$$\|(\rho_0^\gamma)_x\|_{L_x^2} < \infty. \quad (2.63)$$

Lemma 9 *For any $T > 0$, there are constants $C > 0$ depending on α, β , and $C_T > 0$ depending on T such that for $0 \leq t \leq T$,*

$$C \leq \frac{r}{x} \leq Ce^{Ct}, \quad Ce^{-Ct} \leq r_x \leq Ce^{Ct}, \quad (2.64)$$

$$\int_0^T dt \int \left(x^2 v_x^2 + x^2 v_{xt}^2 + v^2 + v_t^2 + v_{xt}^2 + \frac{v_t^2}{x^2} \right) dx \leq C_T, \quad (2.65)$$

$$\int \left(v_x^2 + \frac{v^2}{x^2} + \rho_0 v_t^2 \right) dx \leq C_T, \quad (2.66)$$

and therefore,

$$\int v^2 dx \leq C_T. \quad (2.67)$$

Proof (2.64) is a direct consequence of (2.1), (2.2). (2.65) and (2.66) follow from (2.4), (2.7), (2.28), (2.33). Embedding theory yields (2.67). \square

Next lemma is concerning the regularity at the centre inspired by [23].

Lemma 10 *There is a constant $C_T > 0$ such that,*

$$\int r_{xx}^2 dx + \int \left[\left(\frac{r}{x} \right)_x \right]^2 dx \leq C_T, \quad (2.68)$$

$$\int v_{xx}^2 dx + \int \left[\left(\frac{v}{x} \right)_x \right]^2 dx \leq C_T. \quad (2.69)$$

Proof Here, we use the structure from [23] to obtain high-order estimates. Define

$$\mathcal{G} = \ln \left(\frac{r^2 r_x}{x^2} \right). \quad (2.70)$$

(1.7) can be written as

$$(2\mu + \lambda) \mathcal{G}_{xt} + \gamma \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \mathcal{G}_x = \left(\frac{x}{r} \right)^2 \rho_0 v_t + \left(\frac{x^2}{r^2 r_x} \right)^\gamma (\rho_0^\gamma)_x. \quad (2.71)$$

Multiple (2.71) with \mathcal{G}_x and integrate the resulting equation in the spatial variable,

$$\begin{aligned} \frac{d}{dt} \left\{ \frac{2\mu + \lambda}{2} \int \mathcal{G}_x^2 dx \right\} + \gamma \int \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \mathcal{G}_x^2 dx &= \int \left(\frac{x}{r} \right)^2 \rho_0 v_t \mathcal{G}_x dx \\ + \int \left(\frac{x^2}{r^2 r_x} \right)^\gamma (\rho_0^\gamma)_x \mathcal{G}_x dx &\leq \int \mathcal{G}_x^2 dx + C \int v_t^2 dx + C \int ((\rho_0^\gamma)_x)^2 dx. \end{aligned}$$

From (2.65), Grönwall's inequality then yields

$$\int \mathcal{G}_x^2 dx \leq C_T, \quad (2.72)$$

and as a consequence of (2.66), (2.71), (2.72),

$$\int \mathcal{G}_{xt}^2 dx \leq C_T. \quad (2.73)$$

Notice

$$\begin{aligned} \mathcal{G}_x &= \frac{x}{r r_x} \left(2r_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{r}{x} \right) \right) \\ \mathcal{G}_{xt} &= \frac{x}{r r_x} \left(2r_x \left(\frac{v}{x} \right)_x + v_{xx} \left(\frac{r}{x} \right) + 2v_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{v}{x} \right) \right) \\ &\quad - \frac{x}{r r_x} \left(\frac{v_x}{r_x} + \frac{v}{r} \right) \left(2r_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{r}{x} \right) \right). \end{aligned}$$

Therefore (2.64), (2.72) implies

$$\begin{aligned} C_T &\geq \int \left(2r_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{r}{x} \right) \right)^2 dx = \underbrace{\int r_{xx}^2 \left(\frac{r}{x} \right)^2 dx}_{(2.74)} \\ &\quad + \underbrace{\int 4r_x r_{xx} \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_x + 4r_x^2 \left[\left(\frac{r}{x} \right)_x \right]^2 dx}_{:= A + B}. \end{aligned}$$

At the same time, we shall use the following identity to manipulate B ,

$$r_x = x \left(\frac{r}{x} \right)_x + \frac{r}{x}, \quad r_{xx} = x \left(\frac{r}{x} \right)_{xx} + 2 \left(\frac{r}{x} \right)_x.$$

Consequently, it follows from a complicated and direct calculation and integration by parts,

$$\begin{aligned}
B &= 4 \int \left\{ \left(x \left(\frac{r}{x} \right)_x + \frac{r}{x} \right) \left(x \left(\frac{r}{x} \right)_{xx} + 2 \left(\frac{r}{x} \right)_x \right) \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_x \right. \\
&\quad \left. + \left(x \left(\frac{r}{x} \right)_x + \frac{r}{x} \right)^2 \left[\left(\frac{r}{x} \right)_x \right]^2 \right\} dx \\
&= \left(2x \left(\frac{r}{x} \right)^2 \left[\left(\frac{r}{x} \right)_x \right]^2 + \frac{4}{3} x^2 \left(\frac{r}{x} \right) \left[\left(\frac{r}{x} \right)_x \right]^3 \right) \Big|_{x=1} + 10 \int \left(\frac{r}{x} \right)^2 \left[\left(\frac{r}{x} \right)_x \right]^2 dx \\
&\quad + \frac{28}{3} \int x \left(\frac{r}{x} \right) \left[\left(\frac{r}{x} \right)_x \right]^3 dx + \frac{8}{3} \int x^2 \left[\left(\frac{r}{x} \right)_x \right]^4 dx \\
&= \left(2x \left(\frac{r}{x} \right)^2 \left[\left(\frac{r}{x} \right)_x \right]^2 + \frac{4}{3} x^2 \left(\frac{r}{x} \right) \left[\left(\frac{r}{x} \right)_x \right]^3 \right) \Big|_{x=1} \\
&\quad + \frac{2}{3} \int \left(2x \left[\left(\frac{r}{x} \right)_x \right]^2 + \frac{7}{2} \frac{r}{x} \left(\frac{r}{x} \right)_x \right)^2 dx + \frac{11}{6} \int \left(\frac{r}{x} \right)^2 \left[\left(\frac{r}{x} \right)_x \right]^2 dx.
\end{aligned} \tag{2.75}$$

Meanwhile, from (2.64)

$$\left| \left(\frac{r}{x} \right)_x \Big|_{x=1} \right| = |(r_x - r)|_{x=1}| \leq C_T.$$

Consequently, from (2.74), it holds,

$$\int \left(\frac{r}{x} \right)^2 r_{xx}^2 dx + \int \left(\frac{r}{x} \right)^2 \left[\left(\frac{r}{x} \right)_x \right]^2 dx + \int x^2 \left[\left(\frac{r}{x} \right)_x \right]^4 dx \leq A+B+C_T \leq C_T, \tag{2.76}$$

and together with (2.64), (2.68) follows. Similarly, from (2.1), (2.64), (2.68), (2.73),

$$C_T \geq C_T \int \mathcal{G}_{xt}^2 dx \geq \int \left(2r_x \left(\frac{v}{x} \right)_x + v_{xx} \left(\frac{r}{x} \right) \right)^2 dx - C_T, \tag{2.77}$$

and hence,

$$\begin{aligned}
C_T &\geq \int \left(2r_x \left(\frac{v}{x} \right)_x + v_{xx} \left(\frac{r}{x} \right) \right)^2 dx = 4 \underbrace{\int r_x^2 \left[\left(\frac{v}{x} \right)_x \right]^2 dx + \int \left(\frac{r}{x} \right)^2 v_{xx}^2 dx}_{\text{}} \\
&\quad + 4 \underbrace{\int r_x \left(\frac{r}{x} \right) \left(\frac{v}{x} \right)_x v_{xx} dx}_{\text{}} := \tilde{A} + \tilde{B}.
\end{aligned} \tag{2.78}$$

Making use of the identity,

$$v_{xx} = x \left(\frac{v}{x} \right)_{xx} + 2 \left(\frac{v}{x} \right)_x,$$

integration by parts yields

$$\begin{aligned} \tilde{B} &= 4 \int r_x \left(\frac{r}{x} \right) \left(\frac{v}{x} \right)_x \left(x \left(\frac{v}{x} \right)_{xx} + 2 \left(\frac{v}{x} \right)_x \right) dx \\ &= \left(2xr_x \left(\frac{r}{x} \right) \left[\left(\frac{v}{x} \right)_x \right]^2 \right) \Big|_{x=1} + 6 \int r_x \left(\frac{r}{x} \right) \left[\left(\frac{v}{x} \right)_x \right]^2 dx \\ &\quad - 2 \int xr_{xx} \left(\frac{r}{x} \right) \left[\left(\frac{v}{x} \right)_x \right]^2 dx - 2 \int xr_x \left(\frac{r}{x} \right)_x \left[\left(\frac{v}{x} \right)_x \right]^2 dx, \end{aligned} \quad (2.79)$$

where

$$\left| \left(\frac{v}{x} \right)_x \Big|_{x=1} \right| = |(v_x - v)|_{x=1}| \leq C_T.$$

Therefore, from (2.78), we shall have

$$\begin{aligned} &\int \left(\frac{r}{x} \right)^2 v_{xx}^2 dx + \int \left(4r_x^2 + 6r_x \left(\frac{r}{x} \right) - 2xr_x \left(\frac{r}{x} \right)_x \right) \left[\left(\frac{v}{x} \right)_x \right]^2 dx \\ &\leq C_T + 2 \int xr_{xx} \left(\frac{r}{x} \right) \left[\left(\frac{v}{x} \right)_x \right]^2 dx \\ &\leq C_T + \int \left(\frac{r}{x} \right)^2 \left[\left(\frac{v}{x} \right)_x \right]^2 dx + C \int r_{xx}^2 \left| x \left(\frac{v}{x} \right)_x \right|^2 dx. \end{aligned} \quad (2.80)$$

Then, since $r_x = \frac{r}{x} + x \left(\frac{r}{x} \right)_x$,

$$\begin{aligned} 4r_x^2 + 6r_x \left(\frac{r}{x} \right) - 2xr_x \left(\frac{r}{x} \right)_x &= 2x^2 \left(\frac{r}{x} \right)_x^2 + 12x \frac{r}{x} \left(\frac{r}{x} \right)_x + 10 \left(\frac{r}{x} \right)^2 \\ &\geq 9 \left(\frac{r}{x} \right)^2 - Cx^2 \left(\frac{r}{x} \right)_x^2. \end{aligned} \quad (2.81)$$

Consequently, from (2.80), it follows

$$\begin{aligned} &\int \left(\frac{r}{x} \right)^2 v_{xx}^2 dx + 8 \int \left(\frac{r}{x} \right)^2 \left[\left(\frac{v}{x} \right)_x \right]^2 dx \leq C_T + C_T \int r_{xx}^2 dx \\ &\quad + C \int x^2 \left(\frac{r}{x} \right)_x^2 \left[\left(\frac{v}{x} \right)_x \right]^2 dx \leq C_T + C_T \left\{ \int r_{xx}^2 dx + \int \left(\frac{r}{x} \right)_x^2 dx \right\}, \end{aligned}$$

where it has been applied (2.1), (2.64) and

$$\left| x \left(\frac{v}{x} \right)_x \right| = \left| v_x - \frac{v}{x} \right| \leq C_T.$$

Therefore

$$\int v_{xx}^2 dx + \int \left[\left(\frac{v}{x} \right)_x \right]^2 dx \leq C_T. \quad (2.82)$$

□

2.4 Higher Regularity

In this section, we discuss the regularity propagated by the classical solution to (1.7). It is assumed (2.1) and

$$\|(\rho_0)_x\|_{L_x^2} < \infty, \quad \|(\rho_0^\gamma)_{xx}\|_{L_x^2} < \infty. \quad (2.83)$$

After taking the time derivative of (2.8), one has

$$\begin{aligned} x^2 \rho_0 v_{ttt} + r^2 \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_{x tt} &= r^2 \mathfrak{B}_{xtt} + 4\mu r^2 \left(\frac{v}{r} \right)_{x tt} \\ &+ 4rv \left(\mathfrak{B}_{xt} + 4\mu \left(\frac{v}{r} \right)_{xt} \right) - 4rv \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_{xt} \\ &+ 2(rv)_t \left(\mathfrak{B}_x + 4\mu \left(\frac{v}{r} \right)_x \right) - 2(rv)_t \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_x, \end{aligned} \quad (2.84)$$

with the additional boundary conditions

$$\left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_{tt} - \mathfrak{B}_{tt} \Big|_{x=1} = 0, \quad v_{tt}(0, t) = 0. \quad (2.85)$$

In the following, unless it is stated otherwise, $C > 0$ is a generic constant depending on $\mu, \lambda, \alpha, \beta, \bar{\rho}_0, \gamma$.

Lemma 11 *There is a constant $C > 0$ such that,*

$$\frac{1}{2} \int x^2 \rho_0 v_{tt}^2 dx + \int \int r^2 r_x \left(\frac{v_{xtt}^2}{r_x^2} + 2 \frac{v_{tt}^2}{r^2} \right) dx \leq C(\mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_3). \quad (2.86)$$

Proof Multiply (2.84) with v_{tt} and integrate the resulting in spatial variable. Similar as before, integration by parts then yields

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int x^2 \rho_0 v_{tt}^2 dx + 2\mu \int r^2 r_x \left(\frac{v_{xtt}^2}{r_x^2} + 2 \frac{v_{tt}^2}{r^2} \right) dx \\ & + \lambda \int r^2 r_x \left(\frac{v_{xtt}}{r_x} + 2 \frac{v_{tt}}{r} \right)^2 dx = L_1 + L_2 + L_3, \end{aligned} \quad (2.87)$$

with

$$\begin{aligned} L_1 &= \int \left(6(2\mu + \lambda) \frac{r v_x v_{xt} v_{tt}}{r_x} + 12(2\mu + \lambda) \frac{r_x v v_t v_{tt}}{r} \right. \\ & + 3(2\mu + \lambda) \frac{r^2 v_x v_{xt} v_{xtt}}{r_x^2} + 6\lambda v v_t v_{xtt} - 4(2\mu + \lambda) \frac{r v_x^3 v_{tt}}{r_x^2} \\ & - 8(\lambda + 3\mu) \frac{r_x v^3 v_{tt}}{r^2} - 2(2\mu + \lambda) \frac{r^2 v_x^3 v_{xtt}}{r_x^3} - 4\lambda \frac{v^3 v_{xtt}}{r} \\ & \left. - 12\mu v v_{xt} v_{tt} - 12\mu v_x v_t v_{tt} + 24\mu \frac{v^2 v_x v_{tt}}{r} \right) dx, \\ L_2 &= \int (r^2 v_{tt})_x \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_{tt} dx, \\ L_3 &= - \int 4(r v v_{tt})_x \left(\mathfrak{B}_t - \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_t \right) dx + 16\mu \int r v \left(\frac{v}{r} \right)_{xt} v_{tt} dx \\ & - \int 2((r v)_t v_{tt})_x \left(\mathfrak{B} - \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right) dx + 8\mu \int (r v)_t \left(\frac{v}{r} \right)_x v_{tt} dx. \end{aligned}$$

Hölder inequality together with (2.1) then yields

$$\begin{aligned} L_1, L_2, L_3 &\leq \epsilon \int r^2 r_x \left(\frac{v_{xtt}^2}{r_x^2} + \frac{v_{tt}^2}{r^2} \right) dx + C_\epsilon \int r^2 r_x \left(\frac{v_{xt}^2}{r_x^2} + \frac{v_t^2}{r^2} \right) dx \\ &+ C_\epsilon \int r^2 r_x \left(\frac{v_x^2}{r_x^2} + \frac{v^2}{r^2} \right) dx \end{aligned} \quad (2.88)$$

From (2.4), (2.7), integration in the temporal variable of (2.87) yields (2.86). \square

Lemma 12 *There is a constant $C > 0$ such that*

$$\int v_{tt}^2|_{x=1} dt \leq C(\mathcal{E}_0 + \mathcal{E}_1 + \mathcal{E}_3). \quad (2.89)$$

Proof It follows from similar arguments as in Lemma 3. Proof is omitted here. \square

Lemma 13 *There is a polynomial $P = P(y_1, y_2, y_3, y_4, y_5)$ such that,*

$$\frac{1}{2} \int \left(\frac{x}{r}\right)^2 \rho_0 v_{tt}^2 dx + \int \int \left(\frac{v_{xtt}^2}{r_x} + r_x \frac{v_{tt}^2}{r^2}\right) dx dt \leq P(\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3, \mathcal{E}_4). \quad (2.90)$$

In particular, for any $T > 0$, there is a constant $C_T > 0$ such that

$$\int \rho_0 v_{tt}^2 dx + \int_0^T \int \left(v_{xtt}^2 + \frac{v_{tt}^2}{x^2}\right) dx dt \leq C_T, \quad (2.91)$$

$$\int v_{xt}^2 + \left(\frac{v_t}{x}\right)^2 dx \leq C_T. \quad (2.92)$$

Proof Taking the first derivative in the temporal variable of (2.34) yields

$$\begin{aligned} \left(\frac{x}{r}\right)^2 \rho_0 v_{ttt} + \left[\left(\frac{x^2 \rho_0}{r^2 r_x}\right)^\gamma\right]_{xtt} &= (2\mu + \lambda) \left[\frac{(r^2 v)_x}{r^2 r_x}\right]_{xtt} \\ &+ 2 \left(\frac{x^2 v}{r^2 r} \rho_0 v_t\right)_t + 2 \left(\frac{x}{r}\right)^2 \frac{v}{r} \rho_0 v_{tt}. \end{aligned} \quad (2.93)$$

Multiply (2.93) with v_{tt} and integrate the resulting in the spatial variable. It follows after integration by parts that

$$\frac{d}{dt} \frac{1}{2} \int \left(\frac{x}{r}\right)^2 \rho_0 v_{tt}^2 dx + (2\mu + \lambda) \int \left(\frac{v_{xtt}^2}{r_x} + r_x \frac{v_{tt}^2}{r^2}\right) dx = L_1 + L_2 + L_3 + L_4 + L_5. \quad (2.94)$$

with

$$\begin{aligned} L_1 &= \int \left(\frac{x}{r}\right)^2 \frac{v}{r} \rho_0 v_{tt}^2 dx, \\ L_2 &= (2\mu + \lambda) \int \left(-2 \left(2 \frac{v^3}{r^3} v_{xtt} + \frac{v_x^3}{r_x^3} v_{xtt}\right) + 3 \left(2 \frac{v_t v v_{xtt}}{r^2} + \frac{v_x v_{xt} v_{xtt}}{r_x^2}\right)\right) dx, \\ L_3 &= \int v_{xtt} \left[\left(\frac{x^2 \rho_0}{r^2 r_x}\right)^\gamma\right]_{tt} dx, \quad L_4 = 2 \int \left(\frac{x^2 v}{r^2 r} \rho_0 v_t\right)_t v_{tt} dx, \\ L_5 &= -(2\mu + \lambda) \frac{v_{tt}^2}{r} \Big|_{x=1} + 4\mu \left(\frac{v}{r}\right)_{tt} v_{tt} \Big|_{x=1}. \end{aligned}$$

Meanwhile,

$$L_4 = 2 \int \frac{x^2}{r^2} \frac{v}{r} \rho_0 v_{tt}^2 dx - 6 \int \frac{x^2}{r^2} \frac{v^2}{r^2} \rho_0 v_t v_{tt} dx + 2 \int \frac{x^2}{r^2} \rho_0 \frac{v_t^2}{r} v_{tt} dx. \quad (2.95)$$

Additionally, from (1.7),

$$\begin{aligned} \int \frac{x^2}{r^2} \rho_0 \frac{v_t^2}{r} v_{tt} dx &= \int \frac{v_t v_{tt}}{r} \left[(2\mu + \lambda) \left(\frac{(r^2 v)_x}{r^2 r_x} \right)_x - \left(\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right)_x \right] dx \\ &= \int \left(\frac{v_t v_{tt}}{r} \right)_x \cdot A dx + 4\mu \frac{v v_t v_{tt}}{r^2} \Big|_{x=1} \\ &= \int \left(\frac{v_{xt} v_{tt}}{r} + \frac{v_t v_{xtt}}{r} - r_x \frac{v_t v_{tt}}{r^2} \right) \cdot A dx + 4\mu \frac{v v_t v_{tt}}{r^2} \Big|_{x=1} \end{aligned} \quad (2.96)$$

where, from (2.1),

$$\begin{aligned} A &= -(2\mu + \lambda) \left(\frac{v_x}{r_x} + 2 \frac{v}{r} \right) + \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \leq C, \\ 4\mu \frac{v v_t v_{tt}}{r^2} \Big|_{x=1} &\leq C (v_t^2 + v_{tt}^2) \Big|_{x=1} \end{aligned}$$

Therefore, L_1, L_2, L_3, L_4 can be bounded by the following,

$$\begin{aligned} &\delta \left\{ \int r_x \frac{v_{tt}^2}{r^2} dx + \int \frac{v_{xtt}^2}{r_x} dx \right\} + C_\delta \left\{ \int r_x \frac{v_t^2}{r^2} dx + \int \frac{v_{xt}^2}{r_x} dx + \int r_x v_{tt}^2 dx \right. \\ &\quad \left. + \int r_x v_t^2 dx \right\} + \left\{ \int r_x \frac{v^2}{r^2} dx + \int \frac{v_x^2}{r_x} dx \right\} \left\{ \left\| \frac{v}{r} \right\|_{L_x^\infty}^2 + \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}^2 \right. \\ &\quad \left. + \left\| \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^{2\gamma} \right\|_{L_x^\infty} \right\} + C (v_t^2 + v_{tt}^2) \Big|_{x=1} \end{aligned}$$

Meanwhile, from (2.15),

$$L_5 \leq C (v^2 + v_t^2 + v_{tt}^2) \Big|_{x=1}.$$

Hence, integration in temporal variable of (2.94), together with (2.4), (2.7), (2.20), (2.22), (2.28), (2.33), (2.86), (2.13), (2.14), and (2.89) yields (2.90). (2.92) and (2.91) are consequence of (2.90), (2.64), (2.65).

□

Lemma 14 *There is a constant $C_T > 0$ such that,*

$$\int v_{xxt}^2 dx + \int \left[\left(\frac{v_t}{x} \right)_x \right]^2 dx \leq C_T. \quad (2.97)$$

Proof Taking one temporal derivative of (2.71) yields

$$\begin{aligned} (2\mu + \lambda)\mathcal{G}_{xtt} + \gamma \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \mathcal{G}_{xt} &= \gamma^2 \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \left(\frac{v_x}{r_x} + 2\frac{v}{r} \right) \mathcal{G}_x \\ &+ \left(\frac{x}{r} \right)^2 \rho_0 v_{tt} - 2\frac{x^2}{r^2} \frac{v}{r} \rho_0 v_t - \gamma(\rho_0^\gamma)_x \left(\frac{x^2}{r^2 r_x} \right)^\gamma \left(\frac{v_x}{r_x} + 2\frac{v}{r} \right). \end{aligned} \quad (2.98)$$

By (2.1), (2.64), (2.72), (2.73), (2.90), (2.33), it holds,

$$\int \mathcal{G}_{xtt}^2 dx \leq C_T. \quad (2.99)$$

Meanwhile

$$\mathcal{G}_{xtt} = \frac{x}{rr_x} \left(2r_x \left(\frac{v_t}{x} \right)_x + v_{xt} \left(\frac{r}{x} \right) \right) + l_1 + l_2 + l_3 + l_4 + l_5,$$

with

$$\begin{aligned} l_1 &= \frac{x}{rr_x} \left[2 \left(2v_x \left(\frac{v}{x} \right)_x + v_{xx} \left(\frac{v}{x} \right) \right) + 2v_{xt} \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{v_t}{x} \right) \right], \\ l_2 &= -\frac{x}{rr_x} \left(\frac{v_x}{r_x} + \frac{v}{r} \right) \left[2r_x \left(\frac{v}{x} \right)_x + v_{xx} \left(\frac{r}{x} \right) + 2v_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{v}{x} \right) \right], \\ l_3 &= \frac{x}{rr_x} \left(\frac{v_x}{r_x} + \frac{v}{r} \right)^2 \left[2r_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{r}{x} \right) \right], \\ l_4 &= -\frac{x}{rr_x} \left[\frac{v_{xt}}{r_x} + \frac{v_t}{r} - \left(\left(\frac{v_x}{r_x} \right)^2 + \left(\frac{v}{r} \right)^2 \right) \right] \left[2r_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{r}{x} \right) \right], \\ l_5 &= -\frac{x}{rr_x} \left(\frac{v_x}{r_x} + \frac{v}{r} \right) \left[2v_x \left(\frac{r}{x} \right)_x + v_{xx} \left(\frac{r}{x} \right) + 2r_x \left(\frac{v}{x} \right)_x + r_{xx} \left(\frac{v}{x} \right) \right], \end{aligned}$$

satisfying

$$\begin{aligned} |l_1|, |l_2|, |l_3|, |l_4|, |l_5| &\leq C_T \left(\left(\frac{v}{x} \right)_x + v_{xx} + \left(\frac{r}{x} \right)_x + r_{xx} \right. \\ &\quad \left. + \left(v_{xt} + \frac{v_t}{x} \right) \left(\left(\frac{r}{x} \right)_x + r_{xx} \right) \right). \end{aligned} \quad (2.100)$$

Therefore, from (2.99), (2.68) and (2.69),

$$\int \left(2r_x \left(\frac{v_t}{x} \right)_x + v_{xt} \left(\frac{r}{x} \right) \right)^2 dx \leq C_T + C_T \left(\|v_{xt}\|_{L_x^\infty}^2 + \left\| \frac{v_t}{x} \right\|_{L_x^\infty}^2 \right). \quad (2.101)$$

Meanwhile,

$$\begin{aligned} \int \left(2r_x \left(\frac{v_t}{x} \right)_x + v_{xxt} \left(\frac{r}{x} \right) \right)^2 dx &= \int \left(\frac{r}{x} \right)^2 v_{xxt}^2 dx + 4 \int r_x^2 \left(\frac{v_t}{x} \right)_x^2 dx \\ &\quad + 4 \int r_x \left(\frac{r}{x} \right) \left(\frac{v_t}{x} \right)_x v_{xxt} dx. \end{aligned}$$

Similarly as before, since $v_{xxt} = x \left(\frac{v_t}{x} \right)_{xx} + 2 \left(\frac{v_t}{x} \right)_x$,

$$\begin{aligned} \int r_x \left(\frac{r}{x} \right) \left(\frac{v_t}{x} \right)_x v_{xxt} dx &= \frac{1}{2} \int x r_x \left(\frac{r}{x} \right) \left[\left(\frac{v_t}{x} \right)_x^2 \right]_x dx + 2 \int r_x \left(\frac{r}{x} \right) \left(\frac{v_t}{x} \right)_x^2 dx \\ &= \left(\frac{1}{2} x r_x \left(\frac{r}{x} \right) \left(\frac{v_t}{x} \right)_x^2 \right) \Big|_{x=1} + \frac{3}{2} \int r_x \left(\frac{r}{x} \right) \left(\frac{v_t}{x} \right)_x^2 dx - \frac{1}{2} \int x r_{xx} \left(\frac{r}{x} \right) \left(\frac{v_t}{x} \right)_x^2 dx \\ &\quad - \frac{1}{2} \int x r_x \left(\frac{r}{x} \right)_x \left(\frac{v_t}{x} \right)_x^2 dx. \end{aligned}$$

(2.101) then implies

$$\begin{aligned} &\int \left(\frac{r}{x} \right)^2 v_{xxt}^2 dx + \int \left(4r_x^2 + 6r_x \left(\frac{r}{x} \right) - 2x r_x \left(\frac{r}{x} \right)_x \right) \left(\frac{v_t}{x} \right)_x^2 dx \\ &\leq C_T + 2 \int x r_{xx} \left(\frac{r}{x} \right) \left(\frac{v_t}{x} \right)_x^2 dx + C_T \left(\|v_{xt}\|_{L_x^\infty}^2 + \left\| \frac{v_t}{x} \right\|_{L_x^\infty}^2 \right). \end{aligned} \quad (2.102)$$

Then from (2.81) again, it follows

$$\begin{aligned} &\int \left(\frac{r}{x} \right)^2 v_{xxt}^2 dx + 9 \int \left(\frac{r}{x} \right)^2 \left(\frac{v_t}{x} \right)_x^2 dx \leq C_T + C \int x^2 \left(\frac{r}{x} \right)_x^2 \left(\frac{v_t}{x} \right)_x^2 dx \\ &\quad + 2 \int x r_{xx} \left(\frac{r}{x} \right) \left(\frac{v_t}{x} \right)_x^2 dx + C_T \left(\|v_{xt}\|_{L_x^\infty}^2 + \left\| \frac{v_t}{x} \right\|_{L_x^\infty}^2 \right) \\ &\leq C_T + \delta \int \left(\frac{r}{x} \right)^2 \left(\frac{v_t}{x} \right)_x^2 dx + C_\delta \left\| x \left(\frac{v_t}{x} \right)_x \right\|_{L_x^\infty}^2 \left(\int r_{xx}^2 dx + \int \left(\frac{r}{x} \right)_x^2 dx \right) \\ &\quad + C_T \left(\|v_{xt}\|_{L_x^\infty}^2 + \left\| \frac{v_t}{x} \right\|_{L_x^\infty}^2 \right) \\ &\leq C_T + \delta \int \left(\frac{r}{x} \right)^2 \left(\frac{v_t}{x} \right)_x^2 dx + C_T \left(\|v_{xt}\|_{L_x^\infty}^2 + \left\| \frac{v_t}{x} \right\|_{L_x^\infty}^2 \right), \end{aligned}$$

where the last inequality is due to (2.68) and

$$x \left(\frac{v_t}{x} \right)_x = v_{xt} - \frac{v_t}{x}.$$

Therefore, by suitably small $\delta > 0$,

$$\int \left(\frac{r}{x}\right)^2 v_{xxt}^2 dx + \int \left(\frac{r}{x}\right)^2 \left(\frac{v_t}{x}\right)_x^2 dx \leq C_T + C_T \left(\|v_{xt}\|_{L_x^\infty}^2 + \left\| \frac{v_t}{x} \right\|_{L_x^\infty}^2 \right),$$

or, together with (2.64),

$$\begin{aligned} \int v_{xxt}^2 dx + \int \left(\frac{v_t}{x}\right)_x^2 dx &\leq C_T + C_T \left(\|v_{xt}\|_{L_x^\infty}^2 + \left\| \frac{v_t}{x} \right\|_{L_x^\infty}^2 \right) \\ &\leq (1 + C_\delta) C_T + \delta \left(\int v_{xxt}^2 dx + \int \left(\frac{v_t}{x}\right)_x^2 dx \right), \end{aligned} \quad (2.103)$$

where the last inequality is from (2.92) and

$$f^2(x) \leq \int_0^1 f^2 dx + \int 2ff_x dx \leq \delta \int f_x^2 dx + (1 + C_\delta) \int f^2 dx. \quad (2.104)$$

(2.97) is then obtained after choosing δ small enough. \square

Lemma 15 *There is a constant $C_T > 0$,*

$$\left\| v_t, v_{xt}, \frac{v_t}{x} \right\|_{L_x^\infty}^2 \leq C_T. \quad (2.105)$$

Proof This is a direct consequence of (2.92) (2.97). \square

Lemma 16 *There is a constant $C_T > 0$ such that,*

$$\int r_{xxx}^2 dx + \int \left(\frac{r}{x}\right)_{xx}^2 dx \leq C_T, \quad (2.106)$$

$$\int v_{xxx}^2 dx + \int \left(\frac{v}{x}\right)_{xx}^2 dx. \quad (2.107)$$

Proof Take one spatial derivative of (2.71). The resulting equation is

$$\begin{aligned} (2\mu + \lambda) \mathcal{G}_{xxt} + \gamma \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \mathcal{G}_{xx} &= -2 \frac{x^3}{r^3} \left(\frac{r}{x} \right)_x \rho_0 v_t + \frac{x^2}{r^2} \rho_0 v_{xt} + \frac{x^2}{r^2} v_t (\rho_0)_x \\ &+ \left[\left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \right]_x (-\gamma \mathcal{G}_x + (\rho_0^\gamma)_x) + \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma (\rho_0^\gamma)_{xx}. \end{aligned} \quad (2.108)$$

Then it follows

$$\begin{aligned}
\frac{d}{dt} \int \mathcal{G}_{xx}^2 dx + \int \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \mathcal{G}_{xx}^2 dx &\leq \int \mathcal{G}_{xx}^2 dx + C_T \int \left[\left(\frac{r}{x} \right)_x \right]^2 dx \\
&+ C_T \int v_{xt}^2 dx + C_T \int ((\rho_0)_x)^2 dx + C_T \int ((\rho_0^\gamma)_{xx})^2 dx \\
&+ C_T \int \left[((\rho_0^\gamma)_x)^2 + r_{xx}^2 + \left(\frac{r}{x} \right)_x^2 \right] \cdot (\mathcal{G}_x^2 + (\rho_0^\gamma)_x^2) dx,
\end{aligned} \tag{2.109}$$

where (2.105), (2.1), (2.64) are applied above. Meanwhile, notice

$$\begin{aligned}
((\rho_0^\gamma)_x)^2 &\leq C \left(\int ((\rho_0)_x)^2 dx + \int ((\rho_0^\gamma)_{xx})^2 dx \right), \\
\mathcal{G}_x^2 &\leq C \left(\int \mathcal{G}_x^2 dx + \int \mathcal{G}_{xx}^2 dx \right).
\end{aligned}$$

(2.109) together with (2.68) and (2.92) yields

$$\begin{aligned}
\frac{d}{dt} \int \mathcal{G}_{xx}^2 dx + \int \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma \mathcal{G}_{xx}^2 dx &\leq C_T + C_T \int \mathcal{G}_{xx}^2 dx \\
&+ C_T \left[\int ((\rho_0)_x)^2 dx + \int ((\rho_0^\gamma)_{xx})^2 dx \right].
\end{aligned} \tag{2.110}$$

Grönwall's inequality then shows

$$\int \mathcal{G}_{xx}^2 dx \leq C_T. \tag{2.111}$$

Consequently, from (2.108) it holds,

$$\int \mathcal{G}_{xxt}^2 dx \leq C_T. \tag{2.112}$$

Notice,

$$\begin{aligned}
\mathcal{G}_{xx} &= \frac{x}{r r_x} \left(2r_x \left(\frac{r}{x} \right)_{xx} + r_{xxx} \left(\frac{r}{x} \right) \right) \\
&+ \left(\frac{x}{r r_x} \right) \cdot 3r_{xx} \left(\frac{r}{x} \right)_x + \left(\frac{x}{r r_x} \right)_x \left(2r_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{r}{x} \right) \right),
\end{aligned} \tag{2.113}$$

$$\mathcal{G}_{xxt} = \frac{x}{r r_x} \left(2r_x \left(\frac{v}{x} \right)_{xx} + v_{xxx} \left(\frac{r}{x} \right) \right) + l_1 + l_2 + l_3 + l_4, \tag{2.114}$$

where

$$\begin{aligned}
l_1 &= \frac{x}{rr_x} \left(2v_x \left(\frac{r}{x} \right)_{xx} + r_{xxx} \left(\frac{v}{x} \right) \right) - \frac{x}{rr_x} \left(\frac{v}{r} + \frac{v_x}{r_x} \right) \left(2r_x \left(\frac{r}{x} \right)_{xx} \right. \\
&\quad \left. + r_{xxx} \left(\frac{r}{x} \right) \right), \\
l_2 &= 3 \left(\frac{x}{rr_x} \right) \left(v_{xx} \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{v}{x} \right)_x \right) - 3 \left(\frac{x}{rr_x} \right) \left(\frac{v}{r} + \frac{v_x}{r_x} \right) r_{xx} \left(\frac{r}{x} \right)_x, \\
l_3 &= \left(\frac{x}{rr_x} \right)_x \left(2v_x \left(\frac{r}{x} \right)_x + 2r_x \left(\frac{v}{x} \right)_x + v_{xx} \left(\frac{r}{x} \right) + r_{xx} \left(\frac{v}{x} \right) \right), \\
l_4 &= - \left(\frac{x^2}{r^2 r_x^2} \left(r_x \frac{v}{x} + \frac{r}{x} v_x \right) \right)_x \left(2r_x \left(\frac{r}{x} \right)_x + r_{xx} \left(\frac{r}{x} \right) \right).
\end{aligned} \tag{2.115}$$

Then, (2.111), (2.113), (2.64) and (2.68) yield

$$\begin{aligned}
&\int \left(2r_x \left(\frac{r}{x} \right)_{xx} + r_{xxx} \left(\frac{r}{x} \right) \right)^2 dx \leq C_T + C_T \int \left(r_{xx}^2 + \left(\frac{r}{x} \right)_x^2 \right) \\
&\quad \times \left(r_{xx}^2 + \left(\frac{r}{x} \right)_x^2 \right) dx \leq C_T + C_T \left(\|r_{xx}^2\|_{L_x^\infty} + \left\| \left(\frac{r}{x} \right)_x^2 \right\|_{L_x^\infty} \right) \\
&\leq (1 + C_\delta) C_T + \delta \left(\int r_{xxx}^2 dx + \int \left(\frac{r}{x} \right)_{xx}^2 dx \right)
\end{aligned} \tag{2.116}$$

where the last inequality is due to (2.104). Meanwhile,

$$\begin{aligned}
&\int \left(2r_x \left(\frac{r}{x} \right)_{xx} + r_{xxx} \left(\frac{r}{x} \right) \right)^2 dx = 4 \int r_x^2 \left(\frac{r}{x} \right)_{xx}^2 dx + \int \left(\frac{r}{x} \right)^2 r_{xxx}^2 dx \\
&\quad + 4 \int r_x \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_{xx} r_{xxx} dx,
\end{aligned}$$

in which,

$$\begin{aligned}
&4 \int r_x \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_{xx} r_{xxx} dx = 4 \int r_x \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_{xx} \left(x \left(\frac{r}{x} \right)_{xxx} + 3 \left(\frac{r}{x} \right)_{xx} \right) dx \\
&= 2x r_x \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_{xx}^2 \Big|_{x=1} + 10 \int r_x \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_{xx}^2 dx - 2 \int x r_{xx} \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_{xx}^2 dx \\
&\quad - 2 \int x r_x \left(\frac{r}{x} \right)_x \left(\frac{r}{x} \right)_{xx}^2 dx.
\end{aligned}$$

In the meantime, applying (2.104), the boundary term above admits

$$\begin{aligned} & \left. x r_x \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_{xx}^2 \right|_{x=1} = r r_x (r_{xx} - 2r_x + 2r)^2 \Big|_{x=1} \\ & \leq C_T + C_T \|r_{xx}^2\|_{L_x^\infty} \leq (1 + C_\delta) C_T + \delta \int r_{xxx}^2 dx. \end{aligned}$$

Therefore, from (2.68) and (2.116) it follows

$$\begin{aligned} & \int \left(\frac{r}{x} \right)^2 r_{xxx}^2 dx + \int \left(4r_x^2 + 10r_x \left(\frac{r}{x} \right) - 2x r_x \left(\frac{r}{x} \right)_x \right) \left(\frac{r}{x} \right)_{xx}^2 dx \\ & \leq (1 + C_\delta) C_T + 2 \int x r_{xx} \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_{xx}^2 + \delta \left(\int r_{xxx}^2 dx + \int \left(\frac{r}{x} \right)_{xx}^2 dx \right) \\ & \leq (1 + C_\delta) C_T + C_\delta C_T \left\| x \left(\frac{r}{x} \right)_{xx} \right\|_{L_x^\infty}^2 \int r_{xx}^2 dx \\ & \quad + \delta \left(\int r_{xxx}^2 dx + \int \left(\frac{r}{x} \right)_{xx}^2 dx \right) \\ & \leq (1 + C_\delta) C_T + \delta \left(\int r_{xxx}^2 dx + \int \left(\frac{r}{x} \right)_{xx}^2 dx \right) \end{aligned} \tag{2.117}$$

where in the last inequality it is applied

$$\begin{aligned} & \left\| x \left(\frac{r}{x} \right)_{xx} \right\|_{L_x^\infty}^2 = \left\| r_{xx} - 2 \left(\frac{r}{x} \right)_x \right\|_{L_x^\infty}^2 \leq (1 + C_\delta) C_T \\ & \quad + \delta \int \left(r_{xxx}^2 + \left(\frac{r}{x} \right)_{xx}^2 \right) dx. \end{aligned} \tag{2.118}$$

Again, since

$$\begin{aligned} & 4r_x^2 + 10r_x \left(\frac{r}{x} \right) - 2x r_x \left(\frac{r}{x} \right)_x = 2x^2 \left(\frac{r}{x} \right)_x^2 + 16x \left(\frac{r}{x} \right) \left(\frac{r}{x} \right)_x + 14 \left(\frac{r}{x} \right)^2 \\ & \geq 13 \left(\frac{r}{x} \right)^2 - Cx^2 \left(\frac{r}{x} \right)_x^2, \end{aligned} \tag{2.119}$$

it can be derived from (2.117),

$$\begin{aligned}
& \int \left(\frac{r}{x}\right)^2 r_{xxx}^2 dx + 13 \int \left(\frac{r}{x}\right)^2 \left(\frac{r}{x}\right)_{xx}^2 x \leq (1 + C_\delta) C_T \\
& + \int x^2 \left(\frac{r}{x}\right)_x^2 \left(\frac{r}{x}\right)_{xx}^2 dx + \delta \left(\int r_{xxx}^2 dx + \int \left(\frac{r}{x}\right)_{xx}^2 dx \right) \\
& \leq (1 + C_\delta) C_T + \delta \left(\int r_{xxx}^2 dx + \int \left(\frac{r}{x}\right)_{xx}^2 dx \right),
\end{aligned} \tag{2.120}$$

where (2.118), (2.68) is applied again. Therefore, (2.106) follows by choosing small δ and (2.64). Similarly, from (2.114), (2.115), (2.64), (2.68), (2.69), (2.1), and (2.106), the following estimate holds,

$$\begin{aligned}
& \int \left(2r_x \left(\frac{v}{x}\right)_{xx} + v_{xxx} \left(\frac{r}{x}\right) \right)^2 dx \leq C_T + C_T \left(\int r_{xxx}^2 dx + \int \left(\frac{r}{x}\right)_{xx}^2 dx \right. \\
& \quad \left. + \int \left(v_{xx}^2 + \left(\frac{v}{r}\right)_x^2 + r_{xx}^2 + \left(\frac{r}{x}\right)_x^2 \right) \cdot \left(r_{xx}^2 + \left(\frac{r}{x}\right)_x^2 \right) dx \right) \\
& \leq C_T + C_T \left(\|r_{xx}\|_{L_x^\infty}^2 + \left\| \left(\frac{r}{x}\right)_x \right\|_{L_x^\infty}^2 \right) \\
& \leq C_T + C_T \int \left(r_{xx}^2 + r_{xxx}^2 + \left(\frac{r}{x}\right)_x^2 + \left(\frac{r}{x}\right)_{xx}^2 \right) dx \leq C_T.
\end{aligned} \tag{2.121}$$

Again,

$$\begin{aligned}
& \int \left(2r_x \left(\frac{v}{x}\right)_{xx} + v_{xxx} \left(\frac{r}{x}\right) \right)^2 dx = \int \left(\frac{r}{x}\right)^2 v_{xxx}^2 dx + 4 \int r_x^2 \left(\frac{v}{x}\right)_{xx}^2 dx \\
& + 4 \int r_x \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx} v_{xxx} dx,
\end{aligned}$$

and we shall apply integration by parts as in the following,

$$\begin{aligned}
& \int r_x \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx} v_{xxx} dx = \int r_x \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx} \left(x \left(\frac{v}{x}\right)_{xxx} + 3 \left(\frac{v}{x}\right)_{xx} \right) dx \\
& = \left(\frac{1}{2} x r_x \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx}^2 \right) \Big|_{x=1} + \frac{5}{2} \int r_x \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx}^2 dx \\
& \quad - \frac{1}{2} \int \left(x r_{xx} \left(\frac{r}{x}\right) + x r_x \left(\frac{r}{x}\right)_x \right) \left(\frac{v}{x}\right)_{xx}^2 dx.
\end{aligned}$$

And hence it admits,

$$\begin{aligned} & \int \left(\frac{r}{x}\right)^2 v_{xxx}^2 dx + \int \left(4r_x^2 + 10r_x \frac{r}{x} - 2xr_x \left(\frac{r}{x}\right)_x\right) \left(\frac{v}{x}\right)_{xx}^2 dx \\ & \leq 2 \int xr_{xx} \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx}^2 dx - \left(2xr_x \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx}^2\right) \Big|_{x=1} + C_T, \end{aligned} \quad (2.122)$$

or, by noticing (2.119),

$$\begin{aligned} & \int \left(\frac{r}{x}\right)^2 v_{xxx}^2 dx + 13 \int \left(\frac{r}{x}\right)^2 \left(\frac{v}{x}\right)_{xx}^2 dx \leq C_T + C \int x^2 \left(\frac{r}{x}\right)_x^2 \left(\frac{v}{x}\right)_{xx}^2 dx \\ & \quad + 2 \int xr_{xx} \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx}^2 dx - \left(2xr_x \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx}^2\right) \Big|_{x=1} \\ & \leq (1 + C_\delta)C_T + \delta \int \left(\left(\frac{v}{x}\right)_{xx}^2 + v_{xxx}^2\right) dx, \end{aligned} \quad (2.123)$$

where the last inequality is due to (2.68), (2.69), (2.104) and

$$\begin{aligned} & \left| xr_x \left(\frac{r}{x}\right) \left(\frac{v}{x}\right)_{xx}^2 \right|_{x=1} = r_x r (v_{xx} - 2v_x + 2v)^2 \Big|_{x=1} \leq C_T \|v_{xx}\|_{L_x^\infty} + C_T, \\ & \left\| x \left(\frac{v}{x}\right)_{xx} \right\|_{L_x^\infty}^2 = \left\| v_{xx} - 2 \left(\frac{v}{x}\right)_x \right\|_{L_x^\infty}^2 \leq C \|v_{xx}\|_{L_x^\infty}^2 + C \left\| \left(\frac{v}{x}\right)_x \right\|_{L_x^\infty}^2, \\ & \|v_{xx}\|_{L_x^\infty}^2 + \left\| \left(\frac{v}{x}\right)_x \right\|_{L_x^\infty}^2 \leq C_\delta C_T + \delta \int \left(v_{xxx}^2 + \left(\frac{v}{x}\right)_{xx}^2\right) dx. \end{aligned}$$

Therefore, (2.107) is consequence of (2.123) and (2.64). \square

3 Global Existence

3.1 Functional Framework for the Local Well-posedness Theory

In this section, we shall discuss the appropriate functional framework for the local well-posedness theory for the system consisting the equation (1.7) with the boundary condition (1.8) and the following initial data

$$r(x, 0) = \bar{r}(x), \quad v(x, 0) = \bar{u}_0(x). \quad (3.1)$$

Moreover, the coefficient ρ_0 satisfies the condition (1.10), (1.11). Now we define the energy functionals. Denote

$$\begin{aligned}
\mathfrak{E}_0(t) &= \int x^2 \rho_0 v^2 dx + \int r^2 r_x \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma dx + \int x^2 \rho_0 v_t^2 dx \\
&\quad + \int \chi \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx, \\
\mathfrak{E}_1(t) &= \int x^2 \rho_0 v_{tt}^2 dx + \int \chi \left(\frac{x}{r} \right)^2 \rho_0 v_{tt}^2 dx, \\
\mathfrak{D}_0(t) &= \int \left(r^2 \frac{v_x^2}{r_x} + r_x v^2 + r^2 \frac{v_{xt}^2}{r_x} + r_x v_t^2 \right) dx + \int \chi \left(r_x \frac{v_t^2}{r^2} + \frac{v_{xt}^2}{r_x} \right) dx, \\
\mathfrak{D}_1(t) &= \int \left(r^2 \frac{v_{xtt}^2}{r_x} + r_x v_{tt}^2 \right) dx + \int \chi \left(r_x \frac{v_{tt}^2}{r^2} + \frac{v_{xtt}^2}{r_x} \right) dx,
\end{aligned} \tag{3.2}$$

where χ is a cut-off function satisfying

$$\chi = \begin{cases} 1, & x \in (0, 1/4), \\ 0, & x \in (1/2, R_0), \end{cases}$$

and $-8 \leq \chi' \leq 0$. The following local well-posedness theorem would be applied.

Lemma 17 (Local Well-posedness) *For the equation (1.7) with boundary condition (1.8), there is a strong solution $(r, v) = (r, v)(x, t)$ in $t \in (0, T^*)$, $T^* > 0$ with given initial data $(r, v)(x, 0) = (\bar{r}, \bar{u}_0)(x)$ satisfying*

$$\begin{aligned}
&\int x^2 \rho_0 \bar{u}_0^2 dx + \int \bar{r}^2 \bar{r}_x \left(\frac{x^2 \rho_0}{\bar{r}^2 \bar{r}_x} \right)^\gamma dx + \int x^2 \rho_0 \bar{u}_1^2 dx \\
&\quad + \int \chi \frac{x^2}{\bar{r}^2} \rho_0 \bar{u}_1^2 dx + \int \left[\frac{x^2}{\bar{r}^2 \bar{r}_x} \left(\frac{\bar{r}^2 \bar{r}_x}{x^2} \right)_x \right]^2 dx < \infty,
\end{aligned} \tag{3.3}$$

where

$$\bar{u}_1(x) = \frac{\bar{r}^2}{x^2 \rho_0} \left\{ (2\mu + \lambda) \left[\frac{(\bar{r}^2 \bar{u}_0)_x}{\bar{r}^2 \bar{r}_x} \right]_x - \left[\left(\frac{x^2 \rho_0}{\bar{r}^2 \bar{r}_x} \right)^\gamma \right]_x \right\}. \tag{3.4}$$

(r, v) satisfies

$$\begin{cases} r_x, v_x, \frac{r}{x}, \frac{v}{x} \in L_t^\infty((0, T^*), H_x^1(0, 1)), \\ x\sqrt{\rho_0}v, x\sqrt{\rho_0}v_t, \sqrt{\rho_0}v_t \in L_t^\infty((0, T^*), L_x^2(0, 1)), \\ v, xv_x, v_t, xv_{xt}, \frac{v_t}{x}, v_{xt} \in L_t^2((0, T^*), L_x^2(0, 1)). \end{cases} \tag{3.5}$$

If in addition

$$\begin{aligned} & \|(\rho_0)_x\|_{L_x^2(0,1)}, \|(\rho_0^\gamma)_{xx}\|_{L_x^2(0,1)} < \infty, \\ & \int x^2 \rho_0 \bar{u}_2^2 dx + \int \chi \frac{x^2}{\bar{r}^2} \rho_0 \bar{u}_2^2 dx + \int \left\{ \left[\frac{x^2}{\bar{r}^2 \bar{r}_x} \left(\frac{\bar{r}^2 \bar{r}_x}{x^2} \right) \right]_x \right\}^2 dx < \infty, \end{aligned} \quad (3.6)$$

where

$$\begin{aligned} \bar{u}_2 = & \frac{\bar{r}^2}{x^2 \rho_0} \left\{ (2\mu + \lambda) \left[\frac{(\bar{r}^2 \bar{u}_1)_x}{\bar{r}^2 \bar{r}_x} \right]_x + \gamma \left[\left(\frac{x^2 \rho_0}{\bar{r}^2 \bar{r}_x} \right)^\gamma \left(\frac{(\bar{r}^2 \bar{u}_0)_x}{\bar{r}^2 \bar{r}_x} \right) \right]_x \right\} \\ & - (2\mu + \lambda) \frac{\bar{r}^2}{x^2 \rho_0} \left[\frac{\bar{u}_{0,x}^2}{\bar{r}_x^2} + 2 \frac{\bar{u}_0^2}{\bar{r}^2} \right]_x + 2 \frac{\bar{u}_0}{\bar{r}} \bar{u}_1, \end{aligned} \quad (3.7)$$

there is a smooth solution in $t \in (0, T^{**})$ for some $T^{**} > 0$, (3.5) hold. Moreover, the following regularities would also hold

$$\begin{cases} r_x, v_x, \frac{r}{x}, \frac{v}{x} \in L_t^\infty((0, T^{**}], H_x^2(0, 1)), \\ x \sqrt{\rho_0} v_{tt}, \sqrt{\rho_0} v_{tt}, v_{xxt}, \left(\frac{v_t}{x} \right)_x \in L_t^\infty((0, T^{**}], L_x^2(0, 1)), \\ v_{tt}, x v_{xtt}, \frac{v_{tt}}{x}, v_{xtt} \in L_t^2((0, T^{**}], L_x^2(0, 1)). \end{cases} \quad (3.8)$$

In particular,

$$\frac{x^2}{r^2 \bar{r}_x}, \frac{v}{x}, v_x \in \mathcal{C}([0, T^{**}] \times (0, 1)). \quad (3.9)$$

Sketch of proof Here we present the local a priori estimate. The local well-posedness would follow from a similar iteration argument as in [13] or difference arguments as in [23, Appendix A]. Let $P(\cdot)$ be the generic polynomial which might be different from line to line in the following. Also, for convenience, denote

$$M(t) = \sup_{0 \leq s \leq t} \left\{ \left\| \frac{r}{x} \right\|_{L^\infty(\Omega)_x}, \|r_x\|_{L^\infty(\Omega)_x}, \left\| \frac{x}{r} \right\|_{L^\infty(\Omega)_x}, \left\| \frac{1}{r_x} \right\|_{L^\infty(\Omega)_x}, \left\| \frac{v}{x} \right\|_{L^\infty(\Omega)_x}, \|v_x\|_{L^\infty(\Omega)_x} \right\}.$$

The initial data $\partial_t v(x, 0) = \bar{u}_1(x)$, $\partial_t^2 v(x, 0) = \bar{u}_2(x)$ is given by (3.4), (3.7). We claim first and prove later, $\exists \bar{T} > 0$, any $0 \leq t < \bar{T}$,

$$M(t) \leq C_{\bar{T}} \sup_{0 \leq s \leq t} \mathfrak{E}_0(s). \quad (3.10)$$

Multiply (2.34) with χv_t . After integration by parts in the resulting, similar arguments as before would yield,

$$\begin{aligned} & \frac{d}{dt} \frac{1}{2} \int \chi \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + \int \chi \left(r_x \frac{v_t^2}{r^2} + \frac{v_{xt}^2}{r_x} \right) dx \\ & \leq P(M(t)) \times \left\{ \int \chi \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + \int \chi r_x \frac{v^2}{r^2} dx + \int \chi \frac{v_x^2}{r_x} dx \right. \\ & \quad \left. + \int x^2 \rho_0 v_t^2 dx + \int r^2 \frac{v_x^2}{r_x} dx + \int r_x v^2 dx \right\}. \end{aligned}$$

Therefore, together with (2.4) and (2.11), we shall have

$$\begin{aligned} \frac{d}{dt} \mathfrak{E}_0(t) + \mathfrak{D}_0(t) & \leq P(M(t)) \times \left\{ \mathfrak{E}_0(t) + \int r^2 \frac{v_x^2}{r_x} dx \right. \\ & \quad \left. + \int r_x v^2 dx + \int \chi r_x \frac{v^2}{r^2} dx + \int \chi \frac{v_x^2}{r_x} dx \right\}. \end{aligned} \quad (3.11)$$

In the meantime, from (2.5), after integration by parts, it should hold

$$\int r^2 \frac{v_x^2}{r_x} dx + \int r_x v^2 dx \leq P(M(t)) \times \mathfrak{E}_0(t).$$

Similarly, multiplying (1.7) with χv would eventually yield

$$\begin{aligned} \int \chi r_x \frac{v^2}{r^2} dx + \int \chi \frac{v_x^2}{r_x} dx & \leq P(M(t)) \times \left\{ \int r^2 \frac{v_x^2}{r_x} dx + \int r_x v^2 dx \right. \\ & \quad \left. + \int r^2 r_x \left(\frac{x^2 \rho_0}{r^2 r_x} \right)^\gamma dx + \int \chi \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + 1 \right\}. \end{aligned} \quad (3.12)$$

Then (3.11) can be written as

$$\frac{d}{dt} \mathfrak{E}_0(t) + \mathfrak{D}_0(t) \leq P(M(t)) \times \{ \mathfrak{E}_0(t) + 1 \}.$$

Together with (3.10), then it can be shown there is a $0 < T^* < \tilde{T}$ such that

$$\sup_{0 \leq t \leq T^*} \mathfrak{E}_0(t) \leq 2\mathfrak{E}_0(0), \quad (3.13)$$

and

$$\int_0^{T^*} \mathfrak{D}_0(t) dt \leq 2\mathfrak{E}_0(0). \quad (3.14)$$

The regularities follow from similar arguments as in Section 2.3. It should be noted when applying the arguments as in Lemma 10, the initial data for the ODE of $\int \mathcal{G}_x^2 dx$ is given by the last integral in (3.3). Now we briefly demonstrate (3.10). Denote

$$L(t) = \sup_{0 \leq s \leq t} \left\{ \left\| \frac{r}{x} \right\|_{L^\infty(\Omega)_x}, \|r_x\|_{L^\infty(\Omega)_x}, \left\| \frac{x}{r} \right\|_{L^\infty(\Omega)_x}, \left\| \frac{1}{r_x} \right\|_{L^\infty(\Omega)_x} \right\}.$$

Indeed, from (2.42) and (2.44),

$$\begin{aligned} \left| \frac{v_x}{r_x} \right|^2, \left| \frac{v}{r} \right|^2 &\leq H(L) \times \left\{ \int \chi \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + \int x^2 \rho_0 v_t^2 dx + 1 + v^2|_{x=1} \right\} \\ &\leq H(L) \times \left\{ \int \chi \left(\frac{x}{r} \right)^2 \rho_0 v_t^2 dx + \int x^2 \rho_0 v_t^2 dx + \int \left(r^2 \frac{v_x^2}{r_x} + r_x v^2 \right) dx + 1 \right\}, \end{aligned}$$

where $H(L)$ is a function of L and smooth when $L \neq 0$. Meanwhile, from (2.5) again, after integration by parts, it can be derived,

$$\begin{aligned} \int \left(r^2 \frac{v_x^2}{r_x} + r_x v^2 \right) dx &\leq C \int x^2 \rho_0 v^2 dx + C \int x^2 \rho_0 v_t^2 dx \\ &+ \delta \left\{ \int r^2 \frac{v_x^2}{r_x} dx + \int r_x v^2 dx \right\} + C_\delta H(L). \end{aligned}$$

Hence, the following inequality holds,

$$|v_x|, \left| \frac{v}{x} \right| \leq H(L) \times \{ \mathfrak{E}_0(t) + 1 \},$$

from which, together with the fact $\partial_t r = v$, $\partial_t r_x = v_x$, (3.10) follows. Thus we finish the local estimate for the strong solution. As for the smooth solution, multiply (2.93) with χv_{tt} and integrate the resulting. After integration by parts, it shall hold

$$\begin{aligned} \frac{d}{dt} \frac{1}{2} \int \chi \left(\frac{x}{r} \right)^2 \rho_0 v_{tt}^2 dx + \int \chi \left(\frac{v_{xtt}^2}{r_x} + r_x \frac{v_{tt}^2}{r^2} \right) dx \\ \leq P(M(t)) \times \left\{ \mathfrak{E}_0 + \mathfrak{E}_1 + \mathfrak{D}_0 + \int \chi \left(r_x \frac{v^2}{r^2} + \frac{v_x^2}{r_x} \right) dx \right\}. \end{aligned}$$

Together with (2.87), (3.12),

$$\frac{d}{dt} \mathfrak{E}_1 + \mathfrak{D}_1 \leq P(M(t)) \times \{ \mathfrak{E}_0 + \mathfrak{E}_1 + \mathfrak{D}_0 + 1 \}.$$

Therefore, by noticing (3.10), (3.13) and (3.14), $\exists T^{**} \leq T^*$ such that

$$\sup_{0 \leq t \leq T^{**}} \mathfrak{E}_1(t) \leq 3\mathfrak{E}_0(0) + 2\mathfrak{E}_1(0). \quad (3.15)$$

The regularities would follow from the arguments as in Section 2.4. It should be noted when applying the arguments as in Lemma 16, the initial data for the ODE of $\int \mathcal{G}_{xx}^2 dx$ is given by the last integral in (3.6). In particular, (3.9) would be justified. \square

3.2 Global Existence of Smooth solutions

Proof of the Global Existence of Smooth Solutions,

Given initial data $(\rho_0, u_0) = (\rho_0, u_0)(x)$ satisfying (1.10), (1.11), (1.15), (1.16), (1.20) and

$$\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 < \bar{\epsilon}, \quad \mathcal{E}_3, \mathcal{E}_4 < \infty,$$

from (2.54), $\exists \bar{\alpha} \in (1, \infty)$ such that $\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 < \epsilon(\bar{\alpha})$. Moreover, let $\bar{\beta}_0 = \beta_0(\bar{\alpha}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, M)$, $\bar{\beta}_1 = \beta_1(\bar{\alpha}, \mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2, M)$ be given in Lemma 8. From Lemma 17, there is a smooth solution to (1.7) with (1.8) in $(x, t) \in (0, 1) \times (0, \delta_*)$, $\delta_* > 0$. In particular, the regularities listed in (1.21) hold. Moreover, the continuity in (3.9) implies

$$0 < \frac{x^2}{r^2 r_x} \leq \bar{\alpha}^3, \quad 0 \leq \left| \frac{v}{r} \right|, \left| \frac{v_x}{r_x} \right| \leq \bar{\beta}_0 \leq \bar{\beta}_1. \quad (3.16)$$

Denote the existence time as

$$T_* = \sup \{T \geq \delta_* \mid \text{The smooth solution exists in } t \in (0, T) \text{ and (3.16) holds}\}.$$

We claim $T_* = \infty$. Otherwise, $T_* < \infty$. Then from (2.51), (2.52) in Section 2,

$$\left\| \frac{x^2}{r^2 r_x} \right\|_{L_x^\infty}(T_*) < \bar{\alpha}^3, \quad \left\| \frac{v_x}{r_x} \right\|_{L_x^\infty}(T_*) < \bar{\beta}_0, \quad \left\| \frac{v}{r} \right\|_{L_x^\infty}(T_*) < \bar{\beta}_0. \quad (3.17)$$

Moreover, by choosing $r(x, T_*)$, $u(x, T_*)$ as initial data in (3.1) to the problem (1.7) with the boundary condition (1.8), from the a priori estimates in Section 2, it would satisfy (3.3) and (3.6). Therefore, Lemma 17 implies there is $\delta > 0$ such that there is a smooth solution in $t \in [T_*, T_* + \delta)$. Additionally, (3.9) implies that (r, v) satisfies (3.16) for δ small enough. This contradicts

the definition of T_* . This finishes the proof.

Proof of the Global Existence of Strong Solutions

Consider (1.7), (1.8), (1.9) with (ρ_0, u_0) satisfying (1.10), (1.11), (1.15), (1.16) and

$$\mathcal{E}_0, \mathcal{E}_1, \mathcal{E}_2 < \bar{\epsilon}. \quad (3.18)$$

Construct a smooth sequence $(\rho_{0,\iota}, u_{0,\iota}) \in C^\infty$ satisfying (1.10), (1.11), (1.15), (1.16), (1.17) (1.20), and as $\iota \rightarrow 0$,

$$\begin{aligned} \|\rho_{0,\iota}^\gamma - \rho_0^\gamma\|_{H_x^1(0,1)} &\rightarrow 0, & \|x\rho_{0,\iota}^{\gamma/2} - x\rho_0^{\gamma/2}\|_{L^2(0,1)} &\rightarrow 0, \\ \|x\sqrt{\rho_{0,\iota}}u_{0,\iota} - x\sqrt{\rho_0}u_0\|_{L_x^2(0,1)} &\rightarrow 0, & \|x\sqrt{\rho_{0,\iota}}u_{1,\iota} - x\sqrt{\rho_0}u_1\|_{L_x^2(0,1)} &\rightarrow 0, \\ \|\sqrt{\rho_{0,\iota}}u_{1,\iota} - \sqrt{\rho_0}u_1\|_{L_x^2(0,1)} &\rightarrow 0, \end{aligned} \quad (3.19)$$

where u_1 is defined as (1.13) and

$$u_{1,\iota} = \frac{1}{\rho_{0,\iota}} \left\{ (2\mu + \lambda) \left(\frac{(x^2 u_{0,\iota})_x}{x^2} \right)_x - (\rho_{0,\iota}^\gamma)_x \right\}. \quad (3.20)$$

Moreover $(\rho_{0,\iota}, u_{0,\iota})$ would admit the conditions listed in Lemma 17 for the smooth solution to exist. Then for any fixed $T > 0$, there is a smooth solution (r_ι, v_ι) for $t \in (0, T)$ satisfying (3.16) with some $\bar{\alpha}, \bar{\beta}_0 > M$. Notice, from our choice of the approximation sequence, the estimates in Sections 2.1, 2.2, 2.3 are independent of ι . In particular, the following regularities hold regardless of ι ,

$$\begin{aligned} &\|x\sqrt{\rho_{0,\iota}}v_\iota\|_{L_t^\infty((0,T),L_x^2(0,1))}, \|x\sqrt{\rho_{0,\iota}}v_{\iota,t}\|_{L_t^\infty((0,T),L_x^2(0,1))} \leq C_T, \\ &\|\sqrt{\rho_{0,\iota}}v_{\iota,t}\|_{L_t^\infty((0,T),L_x^2(0,1))}, \|v_\iota\|_{L_t^\infty((0,T),H_x^2(0,1))}, \left\| \frac{v_\iota}{x} \right\|_{L_t^\infty((0,T),H_x^1(0,1))} \leq C_T, \\ &\|xv_{\iota,x}\|_{L_t^2((0,T),L_x^2(0,1))}, \|xv_{\iota,xt}\|_{L_t^2((0,T),L_x^2(0,1))}, \|v_\iota\|_{L_t^2((0,T),L_x^2(0,1))} \leq C_T, \\ &\|v_{\iota,t}\|_{L_t^2((0,T),L_x^2(0,1))}, \|v_{\iota,xt}\|_{L_t^2((0,T),L_x^2(0,1))}, \left\| \frac{v_{\iota,t}}{x} \right\|_{L_t^2((0,T),L_x^2(0,1))} \leq C_T, \\ &0 < \frac{x^2}{r_\iota^2 r_{\iota,x}} \leq \bar{\alpha}^3, \quad \left\| \frac{v_\iota}{r_\iota} \right\|_{L_x^\infty(0,1)}, \left\| \frac{v_{\iota,x}}{r_{\iota,x}} \right\|_{L_x^\infty(0,1)} \leq \bar{\beta}_0. \end{aligned}$$

Then, by taking $\iota \rightarrow 0$, standard compactness arguments would yield

$$(r_\iota, v_\iota) \rightarrow (r, v) \text{ strongly in } C((0, T), H_x^1(0, 1)),$$

and (r, v) is a strong solution to (1.7) in $t \in (0, T)$ and the regularities listed in (1.19) hold. This finishes the proof.

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